An Analytical Framework for Public Debt Management*

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Abstract

This paper introduces a macro-finance framework for evaluating the performance of public debt management strategies. Within the model, risk-averse investors may lose confidence in debt sustainability when indebtedness reaches high levels. Since the government's choice of securities affects debt dynamics, public debt management influences default probability in the model. Featuring stochastic macroeconomic shocks, the model produces realistic macro-finance dynamics. It is calibrated and used to explore the cost and risk implications of issuance strategies that vary across three dimensions: maturity, inflation indexation, and GDP indexation.

Keywords: Sovereign Credit Risk, Public Debt Management, Term Structure Models.
JEL: E43, H63, G12

1. Introduction

This paper proposes an analytical framework designed to explore how the structure of government debt affects debt sustainability. Within the model, the debt structure is characterized by three main features: average maturity, indexation to inflation, and indexation to real Gross Domestic Product (GDP). The pricing of government securities is influenced by investors' risk preferences, who typically require an average excess return on securities that perform worse in economic downturns (recessions). Investors also consider high debt levels to be unsustainable, and these concerns are reflected in government bond prices. We employ approximate, grid-based solution methods to address the fixed-point problem arising from the feedback loop between debt accumulation and credit spreads. This solution remains feasible in the context of exogenous stochastic switches in macroeconomic regimes, making the model rich enough to be brought to the data. We demon-

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strate how to utilize the model to assess the effectiveness of various issuance strategies, focusing on their ability to prevent debt-to-GDP ratios from reaching levels that could undermine the government's credibility in fully repaying its debt.

This paper aims to achieve a unique balance between integrated economic modeling and the model's applicability to real-world data. Practitioner-oriented simulation frameworks typically focus on fitting the model to historical data but often comprise distinct components, such as a macroeconomic block and a yield curve block, that lack interaction, such as feedback from debt levels to funding costs. On the other hand, models in academic studies are often too rigid for practical application with real-world data.[1]

Assessing the effectiveness of debt issuance strategies depends on the joint dynamics of security prices and macroeconomic variables, notably inflation and real output. We show this by comparing cost and risk performance in two stylized economies that are almost identical, with the exception of the correlation between inflation and GDP growth: one economy is demand-driven, while the other is supply-driven. In a demand-driven economy, issuing nominal bonds is highly favorable due to generally lower nominal yields resulting from low term premiums. In contrast, this strategy is less advantageous in a supply-driven economy. This result, in line with Hur, Kondo, and Perri (2018), underscores the importance of basing issuance recommendations on macro-finance models that are sufficiently comprehensive and, in particular, that are able to generate realistic risk premiums. Another example concerns the performance of GDP-linked debt. While simplistic models that disregard risk premiums and focus solely on short horizons indicate that GDP-linked bonds should perform well due to their inherent hedging properties, our framework, consistent with Mouabbi et al. (2024), offers a different perspective. More specifically, our approach takes into account the substantial risk premiums that investors would demand to hold GDP-linked bonds. Hence, a massive issuance of such bonds would lead to higher financing costs for the government, translating into higher debt levels, and therefore to wider credit spreads compared to conventional bonds.

[1]The practical importance of incorporating more complex macroeconomic dynamics and risk preferences than the ones typically used in standard sovereign default models has recently been emphasized by Hur, Kondo, and Perri (2018).

The rest of this paper is organized as follows. Section 3 provides a review of the literature related to this paper. Section 2 presents the modelling framework and Section 4 shows how it can be used to assess the performance of debt issuance strategies. Section 5 concludes. Proofs, estimation details, and additional results are gathered in the appendix.

2. Literature review

This section briefly reviews the literature on public debt management. It is completed by Table 1, which provides details on selected papers.

Pioneering contributions by Barro (1979, 1995) stated that, from the sovereign's perspective, the optimal debt management strategy would involve issuing bonds linked to government expenditure to smooth taxes over time. This would however raise substantial moral hazard, and further research has looked for the possibility to achieve tax smoothing using conventional bonds. Angeletos (2002) demonstrates in particular that the government can almost achieve complete market outcomes through fluctuations in the yield curve, recommending the issuance of long-term debt while investing in short-term assets. However, Faraglia, Marcet, and Scott (2010) argue that significant issues arise when relying solely on non-contingent bonds to achieve such fiscal insurance; they highlight that the recommendations from this approach, such as holding multiples of GDP in short bonds (Buera and Nicolini, 2004), are unrealistic, non-robust, and not practiced by real-world governments. Tax smoothing or fiscal insurance are not the only concepts that have been used in the literature to guide optimal public debt management. Greenwood, Hanson, and Stein (2015), in particular, study optimal government debt maturity in a model where investors derive monetary services from holding riskless short-term securities.

While previous articles focus on the optimal maturity structure of public debt, another strand of the literature emphasizes the role of indexing debt to inflation and real activity. Bohn (1990) and Barro (2003) uses a tax-smoothing objective to assess the optimal composition of public debt with respect to inflation indexation. More recently, Schmid et al. (2023) develop a model where a strategic government issues part of its debt in the form of inflation-indexed bonds to prevent future governments from monetizing debt ex-post.

Several studies, including, Froot, Scharfstein, and Stein (1989), Shiller (1998), Kamstra and Shiller (2009, 2010), and Pienkowski (2017) argue that tying debt repayments to the issuer's GDP performance could mitigate the adverse effects of negative economic growth on debt repayment capabilities. Obstfeld and Peri (1998), along with Borensztein and Mauro (2004), propose the issuance of GDP-linked warrants, a derivative security whose payouts are tied to the economic performance of a sovereign entity. Mouabbi, Renne, and Sahuc (2024) emphasize the potential costs associated with these instruments, noting that bondholders would demand excess returns to offset the increased exposure to recession risks.

Many of the previous studies do not incorporate credit risk into their analyses (see the second column of Table 1). Specifically, research focusing on tax-smoothing objectives typically assumes debt sustainability, thus excluding the need to consider debt levels. However, as noted by Missale (1997, 2012), the ability of governments to obtain insurance is constrained by the necessity to maintain credibility at high debt levels. The simulation approaches implemented by practitioners to evaluate cost and risk measures of debt issuance strategies also usually abstract from potential sovereign credit risk (e.g., Bergstrom et al., 2002; Pick and Anthony, 2008; Bolder and Deeley, 2011; Balibek and Memis, 2012; Bernaschi et al., 2019). In the present paper, we argue that there are performance metrics capable of integrating both tax-smoothing and debt sustainability objectives. For example, a strategy that limits the occurrence of high debt—measured by the upper percentiles of the debt-to-GDP distribution—aligns with both hedging against fiscal shocks and controlling sovereign credit risk.

3. Model

3.1. Overview

We consider an economy populated by a representative risk-averse agent that prices instruments whose payoffs are exposed to the default event of the government. The government default status is denoted by a binary variable $\mathcal{D}_t$, with $\mathcal{D}_t = 1$ if the government has defaulted at or before t, and $\mathcal{D}_t = 0$ otherwise. On date t, the investor observes the default status $\mathcal{D}_t$ as well as new information in the form of a vector z_t; we will often refer

Table 1: Overview of the literature on optimal public debt management

Paper	Default	Model and optimality criteria	Types of Instruments
Greenwood, Hanson, and Stein (2015)	–	3-period model. Investors derive monetary services from holding riskless short-term securities. No inflation. Criterion: Social welfare.	TS-R
Missale and Giavazzi (2005)	–	Simple (i.i.d.) dynamics of inflation, output growth and exchange rate. Criterion: quantiles of debt-to-GDP ratio.	ST-RNX
Debortoli, Nunes, and Yared (2017)	–	Stochastic equilibrium model with fiscal policy distortions. Government cannot commit to fiscal policy. Criterion: social welfare.	ST-N, GD-N
Missale and Blanchard (1994)	–	Study the gov temptation to inflate debt away. Loss function including tax rate	ST-N, LT-N
Drudi and Giordano (2000)	✓	3-period model. Criterion: ad-hoc loss function involving tax rate, inflation, and default costs.	3-period model, ST-RNX, LT-RNX
Angeletos (2002)	–	Stochastic production economy with distortionary taxes. Incomplete markets. Criterion: Social welfare.	ST-R and P-R
Buera and Nicolini (2004)	–	Stochastic production economy with distortionary taxes. Incomplete markets. Criterion: Social welfare.	TS-R
Faraglia, Marcet, and Scott (2010)	–	Stochastic production economy with distortionary taxes. Incomplete markets. Criterion: Social welfare.	TS-N
Bhandari, Evans, Golosov, and Sargent (2017)	–	Stochastic production economy with distortionary taxes. Criterion: Social welfare	ST-R, P-R
Bigio, Nuño, and Passadore (2023)	–	Deterministic model. The government faces liquidity costs during bond auctions; the model also features preferred-habitat investors. Criterion: social welfare.	TS-N
de Lannoy, Bhandari, Evans, Golosov, and Sargent (2022)	–	General stochastic macro-finance models. Criterion: Social welfare	TS-R
Bocola and Dovis (2019)	✓	Small-scale macro-finance model *à la* Cole and Kehoe (2000). Decay coupon rate is time-varying (endogenous). Criterion: Social welfare.	GD-N

Notes: This table reviews some papers belonging to the literature on optimal debt management. Acronyms in the last column are as follows: 'TV' is for time-varying; 'TS' is for term structure; '-R' is for real (referring to inflation-linked securities); '-N' is for nominal (referring to nominal securities); 'ST' is for short-term; 'GD' is for geometrically decaying (referring to perpetuities with geometrically decaying coupons); 'P-' is for perpetuities (or console); 'RNX' refer to bonds that can be either nominal, real (or inflation-indexed), or indexed to a foreign exchange rate).

to the latter as the state vector. This vector includes the Gross Domestic Product (GDP, Y_t), assumed to be totally consumed on date t, a price index (Π_t), the government debt-to-GDP ratio (d_t), the debt service (r_t), and the government budget surplus (s_t). The last three variables are expressed as fractions of GDP. The (log) growth rates of GDP and of the price index are respectively denoted by Δy_t and π_t. (Hence, π_t is the inflation rate.) The latter two variables are driven by an exogenous vector m_t; as in Renne and Pallara (2024), inflation and output growth may also be caused by the government default itself, as explained in Subsection 3.4.2.

The model is built in such a way that $(z_t, \mathcal{D}_t)$ is a Markovian process; this implies that an expectation conditional on the information accumulated until date t—usually represented by the operator $\mathbb{E}_t(.)$—coincides here with $\mathbb{E}(.|\mathcal{D}_t, z_t)$. We will indifferently use these two notations.

3.2. *Debt dynamics*

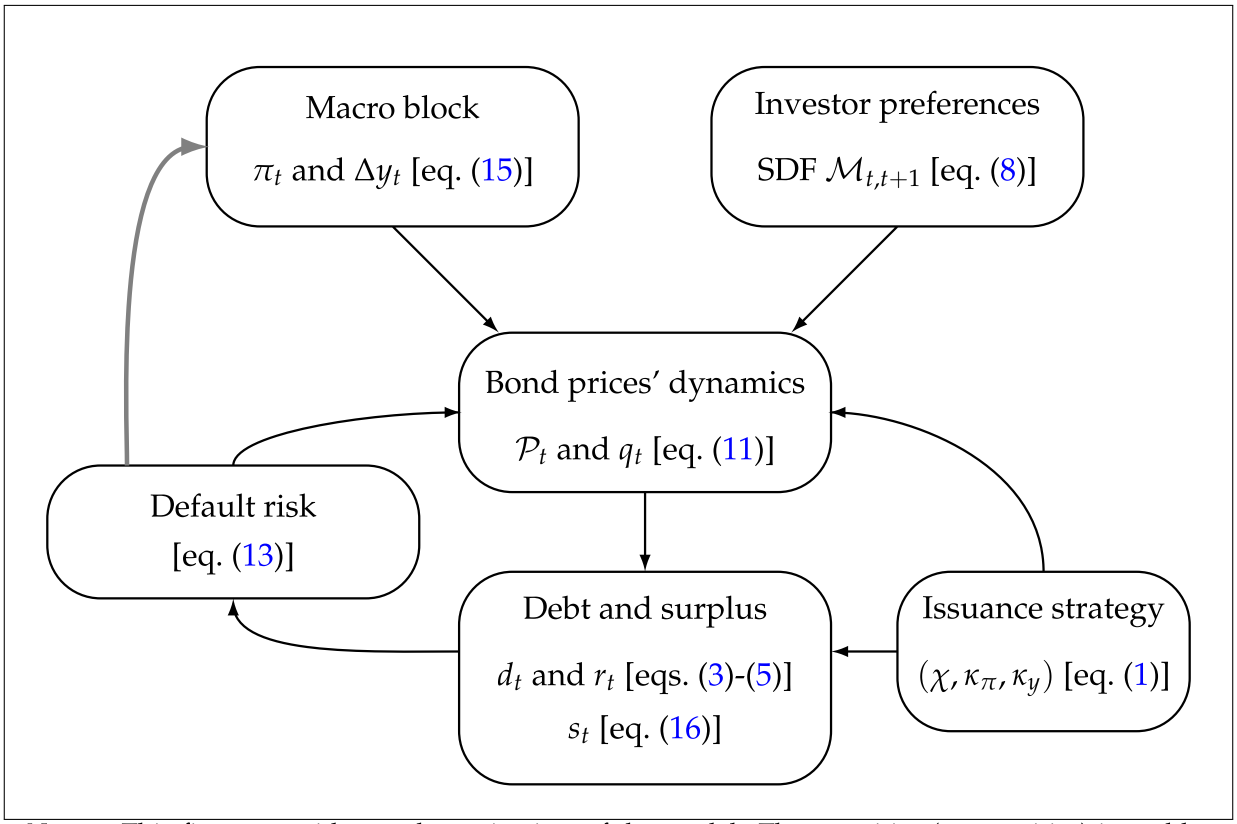
3.2.1. *Government debt issuances*

Following, among others, Leland (1998), Woodford (2001), Hatchondo and Martinez (2009), and Debortoli, Nunes, and Yared (2017), we adopt the simplifying assumption that the government issues perpetuity contracts with coupon payments that decay geometrically at rate χ.[2] Contrary to the previous studies, however, we allow each coupon payment to be indexed to inflation and/or GDP. For that, we construct an index that we call "composite index" as it combines the price index (Π_t) and real GDP (Y_t). Formally, the composite indicator is given by $\Pi_t^{\kappa_\pi} Y_t^{\kappa_y}$, where κ_π and κ_y are indexation coefficients. In the absence of default, a perpetuity issued on date t yields a nominal payment of $\chi^{h-1}\Pi_{t+h}^{\kappa_\pi} Y_{t+h}^{\kappa_y}$ on date $t+h$. Note, for instance, that when $(\kappa_\pi, \kappa_y) = (1, 0)$, the perpetuity provides the equivalent of χ^{h-1} units of goods on date $t+h$—the perpetuity is then perfectly indexed to inflation.[3]

[2]Intuitively, this means that in each period the government issues a set of zero-coupon bonds in constrained relative proportions. The geometrically decaying structure broadly aligns with real-world experience. This assumption is essential for keeping the state-space model manageable; without it, the state would need to encompass not only the total debt d_t but also the entire debt repayment schedule.

[3]The present security is a generalization of the indexed perpetuity proposed by Cochrane (2016). Specifically, Cochrane (2016)'s *indexed perpetuity* corresponds to the present one when $(\chi, \kappa_\pi, \kappa_y) = (1, 0, 1)$.

Figure 1: Schematic representation of the model



Note — This figure provides a schematic view of the model. The securities (perpetuities) issued by the government are described in Subsection 3.2.1; this subsection also defines the price ($\mathcal{P}_t$) and yield-to-maturity (q_t) of these perpetuities (eq. 2). The debt accumulation process is discussed in Subsection 3.2.2. Investor preferences are described in Subsection 3.3.1. The specification of the conditional default probability is given in Subsection 3.4.1. The dynamics of inflation (π_t) and real GDP growth (Δy_t is presented in Subsection 3.4.2. The lower part of the diagram highlights the fixed-point problem inherent in the model: debt dynamics depend on bond prices, which depend on the probability of default, which in turn depends on debt dynamics. The thick grey arrow on the left of the plot represents the potential effect of a default on the macroeconomy (captured by parameters ν_y and ν_π in eq. 15).

The government may default on these perpetuities. Denoting the recovery rate by RR, an investor having purchased the perpetuity on date t receives the following amount on date $t+h$, expressed in currency units:

$$\chi^{h-1}[1 \times (1-\mathcal{D}_{t+h}) + RR \times \mathcal{D}_{t+h}] \times \Pi_{t+h}^{\kappa_\pi} Y_{t+h}^{\kappa_y}. \tag{1}$$

The price of the perpetuity, expressed in units of the composite index, is denoted by $\mathcal{P}_t$. That is, expressed in currency units, the price of the perpetuity $\Pi_t^{\kappa_\pi} Y_t^{\kappa_y} \mathcal{P}_t$. The price $\mathcal{P}_t$ is directly expressed in dollar terms when $(\kappa_\pi, \kappa_y) = (0,0)$; it is expressed in terms of units of goods (i.e., in real terms) when $(\kappa_\pi, \kappa_y) = (1,0)$, and in terms of units of GDP when $(\kappa_\pi, \kappa_y) = (1,1)$.

The perpetuity's yield-to-maturity q_t (or internal rate of return) is defined through:

$$\mathcal{P}_t = \sum_{h=1}^{\infty} \frac{\chi^{h-1}}{(1+q_t)^h} = \frac{1}{1+q_t-\chi}. \tag{2}$$

Note that this return is expressed in terms of composite index units; it coincides in particular with a nominal return when $(\kappa_\pi, \kappa_y) = (0,0)$ and to a real return when $(\kappa_\pi, \kappa_y) = (1,0)$.[4]

3.2.2. Resulting debt accumulation process

Consistently with international debt accounting standards—on which our data are based—the concept of debt valuation we opt for is that of "nominal valuation of debt securities," where the debt outstanding covers the sum of funds originally advanced, plus any subsequent advances, less any repayments, plus any accrued interest.[5]

When the government issues the perpetual bonds presented in Subsection 3.2.1, the debt dynamics is governed by the following equations:

[4]When $(\kappa_\pi, \kappa_y) = (1,1)$, indicating a GDP-linked bond, this return lacks a standard interpretation, as it is neither comparable to a nominal rate nor a real rate. Instead, it must be added to a nominal GDP growth rate in order to be comparable to a nominal yield.

[5]See International Monetary Fund, Bank for International Settlements and European Central Bank (2015). As noted in Renne and Pallara (2024), although such a precision is innocuous in the context of models considering only short-term issuances, it is not in the present context, where the government issues long-dated debt instruments. This debt concept— which is the one used in the Debt Sustainability Analysis (DSA) by investors, market analysts, and international institutions—does not coincide with the market value of debt, see Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2023).

Proposition 1. *In the absence of default until date $t+1$, we have:*

$$
\begin{aligned}
d_{t+1} &= \exp(-\pi_{t+1} - \Delta y_{t+1}) d_t - s_{t+1} + r_{t+1} && (3) \\
r_{t+1} &= \underbrace{\left(\exp(\kappa_\pi \pi_{t+1} + \kappa_y \Delta y_{t+1}) - 1\right) \exp(-\pi_{t+1} - \Delta y_{t+1}) d_t}_{\text{debt indexation } (\zeta_{t+1} - \underline{\zeta}_{t+1}) d_t} + \underline{r}_{t+1} && (4) \\
\underline{r}_{t+1} &= \zeta_{t+1} q(z_t) \underbrace{(d_t - \chi \zeta_t d_{t-1})}_{\text{date-}t\text{ issuances}} + \zeta_{t+1} \chi \underline{r}_t, && (5)
\end{aligned}
$$

where r_t (respectively $\underline{r}_t$) is the debt service including (resp. excluding) indexation costs expressed as a fraction of GDP, and where

$$
\begin{cases}
\zeta_t = \exp[(\kappa_\pi - 1)\pi_t + (\kappa_y - 1)\Delta y_t] \\
\underline{\zeta}_t = \exp(-\pi_t - \Delta y_t).
\end{cases}
$$

The perpetuity's yield-to-maturity q is a function of the state z_t that includes d_t, d_{t-1}, and r_t.

Proof. See Appendix I.1. □

Remarkably, in spite of the fact that, in the current framework, (i) the government may issue long-term securities and (ii) these securities can provide contingent payoffs, the debt accumulation process (3) is similar to simpler models where the government issues only one-period bonds.[6] Indexation and duration however show up in (4) and (5), respectively. Equation (5), in particular, shows that interest payments—excluding indexation—take the form of an exponential smoothing of previous interest rates.

To perfectly stabilize the debt-to-GDP ratio, one would need to have

$$r_t = (1 - \exp(-\pi_t - \Delta y_t)) d_{t-1} + s_t \approx (\pi_t + y_t) d_{t-1} + s_t. \tag{6}$$

To fix ideas, consider the case where $\chi = 0$—the government issues short-term bonds. In that case, if inflation and GDP growth are small, we have, using (4) and (5):

$$r_t \approx (q_{t-1} + \kappa_\pi \pi_t + \kappa_y \Delta y_t) d_{t-1}. \tag{7}$$

[6]In such toy models of debt accumulation, this equation is supplemented by $r_t \approx q_{t-1} d_{t-1}$ if the government issues nominal bonds (q_t then is the short-term nominal yield), and $r_t \approx (q_{t-1} + \pi_t) d_{t-1}$ if the government issues inflation-linked bonds (q_t then is a real rate).

Combining (6) and (7), it comes that the stabilization of d_t between dates $t-1$ and t then requires:

$$s_t = (q_{t-1} + (\kappa_\pi - 1)\pi_t + (\kappa_y - 1)\Delta y_t)d_{t-1}.$$

At date $t-1$, q_{t-1} and d_{t-1} are known. As a result, the government achieves debt stabilization, i.e., $d_t = d_{t-1}$, if $(\kappa_\pi, \kappa_y) = (1,1)$ and if it is able to commit to the future surplus $s_t = q_{t-1}d_{t-1}$. This reasoning is in favor of the issuance of short-term debt indexed to nominal GDP, i.e., $(\chi, \kappa_\pi, \kappa_y) = (0,1,1)$.

The previous reasoning is however subject to several limitations. First, it is difficult for the government to commit to the future surplus $s_t = q_{t-1}d_{t-1}$ in date $t-1$: denoting nominal tax receipts by T_t and government expenditures by G_t, we have $s_t = (T_t - G_t)/(\Pi_t Y_t)$; none of the four terms appearing in the previous expression are easy to forecast from one year to the other. Second, this reasoning focuses on the predictability of the debt-to-GDP ratio from date $t-1$ to date t, i.e. on a single-period horizon. But this predictability does not carry over to longer horizons because, for any type of perpetuity the government issues, q_t randomly varies over time: As of date $t-1$, we know q_{t-1}—that allows to predict d_t—but we do not know q_t—that is needed to forecast d_{t+1}. In other words, if the prices of bonds indexed to GDP are particularly volatile, then medium to long-term forecasts of debt-to-GDP may be worse than that that would result from the issuance of more conventional bonds—even though short-term forecasts are more accurate. Third, this reasoning focuses on the predictability of future debt-to-GDP ratio, abstracting from its level. If investors ask premiums to hold GDP-linked bonds, then the distribution of future debt-to-GDP may be shifted to the right with respect to more standard issuance strategies, reducing the advantage of issuing such securities (Mouabbi et al., 2024).[7]

3.3. Security pricing

This subsection provides general security pricing formulas. Subsection 3.3.1 presents a general SDF specification. Subsection 3.3.2 contains propositions that characterizes the

[7] Even when the distribution of debt-to-GDP exhibits a small variance (e.g., attributable to fiscal insurance mechanisms), a high mean can still result in significant upper quantiles for this distribution. This implies that elevated average funding costs may diminish the advantages of smoothing as far as debt sustainability is concerned (having in mind a fiscal limit concept).

pricing of the government perpetuity and of (indexed) zero-coupon bonds in our economy.

3.3.1. The stochastic discount factor (SDF)

We consider a flexible specification of the stochastic discount factor (SDF):

$$\mathcal{M}^r_{t,t+1} = \exp(f^r(z_t, z_{t+1}) + \nu^r(z_{t+1})\Delta\mathcal{D}_{t+1}). \quad (8)$$

This specification is general and could accommodate various types of time-separable or recursive preferences. In particular, the SDF takes this form when we combine the macroeconomic dynamics described in Subsection 3.4.2 together with Epstein and Zin (1989)'s preferences with unit elasticity (see Appendix II);[8] we will employ such preferences in our quantitative exercises (Section 4).

A particularity of (8) lies in the presence of the sovereign default in the SDF. Intuitively, a SDF reflects economic fears. Consequently, since the SDF of (8) jumps in the event of default when ν^r is positive, this specification captures investors' specific aversion to a government default—for example, because they believe that a government default would generate a crisis that would jeopardize their current and future consumption.[9]

Eq. (8) defines the real SDF. To develop our analysis, we need two other types of SDFs: the nominal one and the one that allows to price payoffs expressed in units of the composite index. It is well-known that the nominal SDF ($\mathcal{M}^\$_{t,t+1}$, say) and the real SDF are linked through $\mathcal{M}^\$_{t,t+1} = \mathcal{M}^r_{t,t+1}\exp(-\pi_{t+1})$. More generally, define $\mathcal{M}_{t,t+1}$ as the SDF that allows to price assets whose payoffs are expressed in units of the composite index; we

[8]Epstein-Zin preferences break the link between relative risk aversion (γ) and the intertemporal elasticity of substitution (set to one here) in CRRA preferences. Bansal and Yaron (2004) and Bansal and Shaliastovich (2013), among others, have demonstrated the appropriateness of such preferences to account for the dynamics of asset prices.

[9]The existence of a direct effect of the default event on the SDF gives rise to specific credit risk premiums, called credit-event risk premiums, in the prices of bonds issued by the considered systemic entity (e.g., Driessen, 2005; Gouriéroux et al., 2014; Bai et al., 2015). Eq. (8) is consistent with the literature studying the asset-pricing influence of disasters (e.g. Barro, 2006, Eq. 7, Arellano, 2008, Eq. 3, Barro and Jin, 2011, Eq. 1, Gabaix, 2012, Eq. 1, Arellano and Ramanarayanan, 2012, last equation of Section III, Wachter, 2013, Eq. 1).

have:[10]

$$\mathcal{M}_{t,t+1} = \mathcal{M}^{\$}_{t,t+1} \frac{\Pi_{t+1}^{\kappa_\pi} Y_{t+1}^{\kappa_y}}{\Pi_t^{\kappa_\pi} Y_t^{\kappa_y}} = \mathcal{M}^{r}_{t,t+1} \exp((\kappa_\pi - 1)\pi_{t+1} + \kappa_y \Delta y_{t+1}).$$

If inflation and output growth admit the following general specifications:

$$\pi_t = f_\pi(z_t) + \nu_\pi \Delta \mathcal{D}_t, \quad \text{and} \quad \Delta y_t = f_y(z_t) + \nu_y \Delta \mathcal{D}_t, \tag{9}$$

we get:

$$\mathcal{M}_{t,t+1} = \exp(f(z_t, z_{t+1}) + \nu(z_{t+1})\Delta \mathcal{D}_{t+1}), \tag{10}$$

where

$$\begin{cases} f(z_t, z_{t+1}) &= f^r(z_t, z_{t+1}) + (\kappa_\pi - 1) f_\pi(z_{t+1}) + \kappa_y f_y(z_{t+1}) \\ \nu(z_{t+1}) &= \nu^r(z_t, z_{t+1}) + (\kappa_\pi - 1)\nu_\pi + \kappa_y \nu_y. \end{cases}$$

3.3.2. Solving for the price of perpetuity and zero-coupon bonds

At this stage, the model is not complete since we have not made explicit the dynamics of inflation, GDP growth and the budget surplus—which are involved in the System (3)-(5)—nor have we expressed the default process (or probability of default). Nevertheless, we can already formulate the restriction that has to be satisfied by the price of the perpetuity issued by the government, whatever the dynamics of (m_t, s_t) or the specification of the government default probability. This is done in Proposition 2.

Proposition 2. *Function q satisfies the following fixed-point problem:*

$$q(z_t) = \chi - 1 + \tag{11}$$

$$\frac{1}{\mathbb{E}_t\left(\exp\left(f(z_t, z_{t+1})\right)\left[\mathcal{D}_{t+1}\left(RRe^{\nu^r(z_{t+1})}(1 + \chi \underline{\mathcal{P}}(z_{t+1})) - \frac{1+q(z_{t+1})}{1+q(z_{t+1})-\chi}\right) + \frac{1+q(z_{t+1})}{1+q(z_{t+1})-\chi}\right]\right)},$$

[10] By definition of $\mathcal{M}_{t,t+1}$, for any asset whose date-t price is x_t (expressed in units of the composite index), we must have: $x_t = \mathbb{E}_t(\mathcal{M}_{t,t+1} x_{t+1})$. But we must also have $x_t Index_t = \mathbb{E}_t(\mathcal{M}^{\$}_{t,t+1} x_{t+1} Index_{t+1})$. We therefore have: $\mathcal{M}_{t,t+1} = \mathcal{M}^{\$}_{t,t+1} Index_{t+1}/Index_t = \mathcal{M}^{\$}_{t,t+1} \exp(\kappa_\pi \pi_{t+1} + \kappa_y \Delta y_{t+1})$. Notice that we have $\mathcal{M}_{t,t+1} = \mathcal{M}^{r}_{t,t+1}$ when $(\kappa_\pi, \kappa_y) = (1, 0)$ since, in this case, the composite-indexed bond is an inflation-linked (or real) bond.

where $\underline{\mathcal{P}}$ is the post-default price of the perpetuity, that is:

$$\underline{\mathcal{P}}(z_t) = \mathbb{E}\left(\sum_{h=1}^{\infty} \chi^{h-1} \mathcal{M}_{t,t+h} \right| \left. \mathcal{D}_t = 1, z_t \right).$$

Proof. See Appendix I.2. □

The solution function $q(.)$ depends on (i) the dynamics of the exogenous macroeconomic factors m_t, (ii) the fiscal rule (i.e., the specification of the budget surplus s_t), and (iii) the conditional probability of default (governing the dynamics of $\mathcal{D}_t$). These ingredients are made precise in Subsection 3.4.

As is the case for the perpetuity, we can also already express general pricing formulas for zero-coupon bonds. Let us first define such bonds and their payoffs. For that, assume that the government has not defaulted before date t (i.e., $\mathcal{D}_t = 0$). Denote by $\mathcal{B}_h(z_t)$ the date-t price of a defaultable zero-coupon bond of maturity h. At maturity, that is on date $t+h$, this bond provides a payoff of 1 if $\mathcal{D}_{t+h} = 0$, and a payoff of $RR < 1$ if $\mathcal{D}_{t+h} = 1$.[11]

Proposition 3. *The prices of zero-coupon bonds can be computed recursively using:*

$$\begin{aligned} \mathcal{B}_h(z_t) \quad = \quad & \mathbb{E}\Big[\exp(f(z_t, z_{t+1}))\mathcal{B}_{h-1}(z_{t+1}) + \\ & \mathcal{D}_{t+1} \exp(f(z_t, z_{t+1})) \left\{ RR e^{\nu(z_{t+1})} \underline{\mathcal{B}}_{h-1}(z_{t+1}) - \mathcal{B}_{h-1}(z_{t+1}) \right\} \Big| \, \mathcal{D}_t = 0, z_t \Big], \end{aligned} \tag{12}$$

starting from $\mathcal{B}_0(x) = 1$ for any state x. In (12), $\underline{\mathcal{B}}_h(z_t)$ denotes the price of a post-default zero-coupon bond, i.e.:

$$\underline{\mathcal{B}}_h(z_t) = \mathbb{E}(\mathcal{M}_{t,t+h} | \mathcal{D}_t = 1, z_t).$$

Proof. See Appendix I.4. □

It is worth mentioning that pricing zero-coupon bond is not necessary to solve for the model since the government issues perpetuities, and not zero-coupon bonds.[12] Neverthe-

[11] This definition coincides with the recovery of market value (RMV) convention of Duffie and Singleton (1999), who consider that, upon default (on date $t+k$, $k \leq h$, say), the recovery payment of a defaultable zero-coupon bond of residual maturity h is equal to a fraction RR of a risk-free zero-coupon bond of maturity equal to the residual maturity of the defaulted bond, i.e., $\mathbb{E}_{t+k}(\mathcal{M}_{t+k,h-k} | \Delta\mathcal{D}_{t+k} = 1)$. Indeed, in the RMV context, the default payoff is $RR\mathbb{E}_{t+k}(\mathcal{M}_{t+k,h-k} | \Delta\mathcal{D}_{t+k} = 1)$, which is also the net present value, as of the default date $(t+k)$, of the payoff RR at the original maturity date $(t+h)$.

[12] The perpetuity can however be seen as a collection of zero-coupon bonds: if $\mathcal{D}_t = 0$, we have $\mathcal{P}(z_t) = \sum_{h=1}^{\infty} \chi^{h-1} \mathcal{B}_h(z_t)$.

less, it can be useful to compute such bond prices and the associated yields-to-maturity; this is for instance the case for model validation, since zero-coupon yields are more standard objects than the perpetuity's yield-to-maturity.

3.4. Other modelling ingredients

In this subsection, we complete the model by introducing (i) dynamics for the exogenous macroeconomic factor m_t and (ii) a specification for the conditional probability of default. We however stress that the results presented above are general and could accommodate alternative specifications for (i) and (ii).

3.4.1. Sovereign default probability

The specification of the conditional probability of default of the government is borrowed from Pallara and Renne (2024). It takes the form of a decreasing function of the fiscal space, $d^* - d_t$, where d^* can be understood as a fiscal limit. If the sovereign default occurs as soon as $d^* > d_t$, the fiscal limit is "strict," in the sense that default is automatically triggered when the limit is breached. However, in order to capture non-modeled factors that may precipitate or delay default—e.g. political factors (Hatchondo and Martinez, 2010)—we introduce a Gaussian white noise ν_t, of variance σ_ν^2, and assume that the probability of default depends on $d^* - d_t + \nu_{t+1}$, which is a noisy measure of the fiscal space. Specifically, the conditional probability of observing a sovereign default on date $t+1$ is of the form:

$$\mathbb{P}(\mathcal{D}_{t+1} = 1 | \mathcal{D}_t = 0, z_t, \nu_{t+1}) = \mathcal{F}(d^* - d_t + \nu_{t+1}), \tag{13}$$

where $\mathcal{F}$ is a function valued in $[0, 1]$. Function $\mathcal{F}$ is such that $\mathcal{F}(u) = 0$ for $u \geq 0$, implying that the default probability is equal to zero as long as the (noisy) fiscal space debt is nonnegative. Moreover, function $\mathcal{F}$ is increasing: the larger the distance between debt and the fiscal limit, the higher the probability of default. In the following, we employ the following specification for $\mathcal{F}$:

$$\mathcal{F}(d_t - d^* - \nu_{t+1}) = 1 - \exp(-\underbrace{\max[0, \alpha(d_t - d^* - \nu_{t+1})]}_{=\underline{\lambda}_{t+1}}), \tag{14}$$

with $\alpha > 0$. $\underline{\lambda}_{t+1} \equiv \alpha \max(0, d_t - d^* - \nu_{t+1})$ is the default intensity. When it is small, it is close to the conditional probability of default $\mathbb{P}(\mathcal{D}_{t+1} = 1 | \mathcal{D}_t = 0, z_t, \nu_{t+1})$.

The parameter α is another modeling ingredient—on top of ν_t—that makes it possible to control for the strictness of the fiscal limit. This is illustrated by Figure 2, that displays the probabilities of observing a default over a year, conditional on a given level of the fiscal space $d_t - d^*$, and for different values of α. If α is large, the fiscal limit is strict, in the sense that default is likely to happen as soon as $d_t > d^*$. By contrast, if α is small, the fiscal limit is softer, in the sense that, for the same value $d_t - d^* > 0$, a sovereign default on date t is possible, but less likely.[13]

Figure 2 illustrates the influence of α and the fiscal space on the probability of default.

3.4.2. Dynamics of macroeconomic variables

We assume that the dynamics of inflation and GDP growth are governed by

$$\pi_t = \mu_\pi' m_t + \nu_\pi \Delta\mathcal{D}_t, \quad \text{and} \quad \Delta y_t = \mu_y' m_t + \nu_y \Delta\mathcal{D}_t, \tag{15}$$

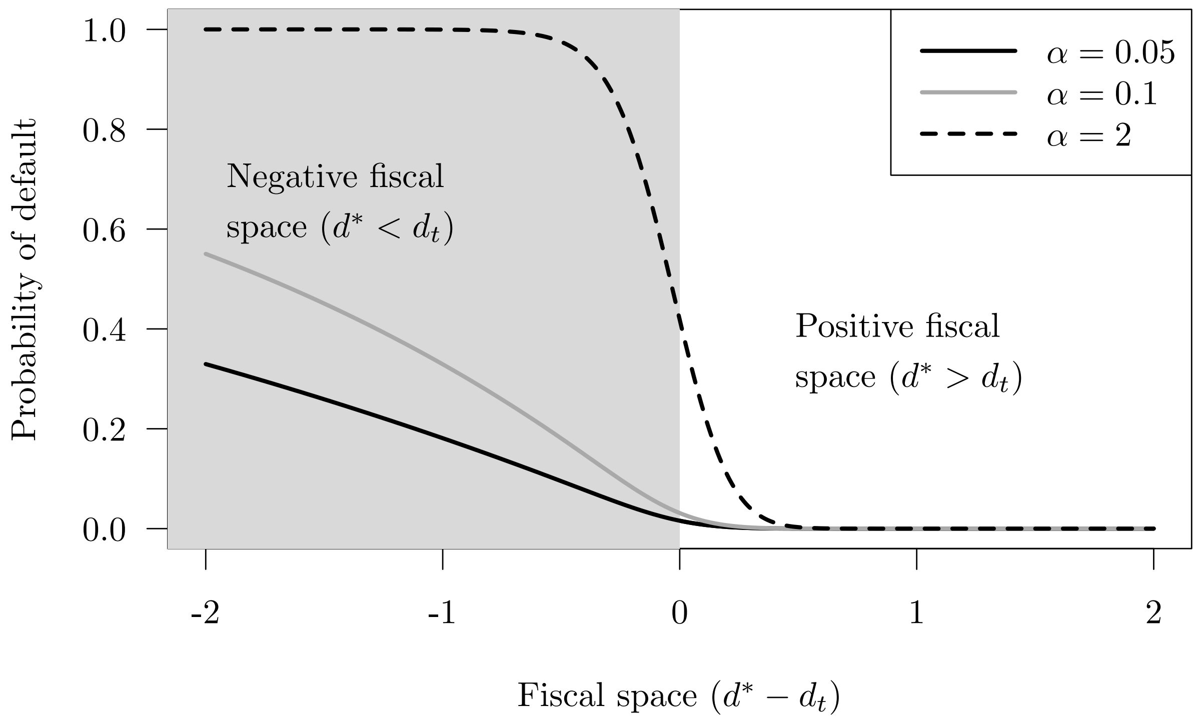
where m_t is a selection vector of dimension $n_m \times 1$. It can be noted that (15) is consistent with (9) since m_t is part of the state vector z_t. Vector m_t follows an exogenous and homogeneous regime-switching process.[14] Specifically, its dynamics is defined through a matrix of transition probabilities Ω. Specifically, this matrix is such that

$$\mathbb{P}(m_{t+1} = e_j | m_t = e_i) = e_i' \Omega e_j,$$

[13]As argued in Pallara and Renne (2024), the notion of soft fiscal limit is consistent with the widespread idea that it is difficult to assess sovereign debt sustainability (e.g., Warmedinger et al., 2017; Debrun et al., 2019), which gives rise to "grey areas" where default becomes likely but can also be avoided. The World Bank and the IMF themselves reckon that, alongside quantitative approaches, the use of judgment is needed to assess sovereign debt sustainability (IMF and World Bank, 2021). The present modelling can also be seen as a way to capture "crisis zones" arising multiple-equilibria frameworks *à la* Cole and Kehoe (2000).

[14]The fact that inflation is driven by exogenous shocks—aside from the endogenous default effect—contrasts with the literature on debt and inflation that emphasizes strategic inflation, whereby governments use inflation to manage high debt burdens during economic downturns (e.g., Missale and Blanchard, 1994). However, as Hur, Kondo, and Perri (2018) note, such mechanisms are more applicable to emerging economies; in advanced economies, greater monetary policy independence and monetary union constraints make them less relevant.

Figure 2: Annual probability of default with respect to fiscal space ($d^* - d_t$)



Note: This figure shows the posited relationship between sovereign default probability and fiscal space ($d^* - d_t$). More precisely, it shows $\mathbb{P}(\mathcal{D}_{t+4} = 1 | \mathcal{D}_t = 0, d^* - d_{t+i} = u, i = 0, \dots, 3)$. According to (13) and (14), conditionally on the fiscal space $d^* - d$ and on the noise ν (with $\nu \sim i.i.d. \mathcal{N}(0, \sigma_\nu^2)$, and $\sigma_\nu = 0.2$ here), the (one-quarter) probability of default is $1 - \exp(-\underline{\lambda})$, where $\underline{\lambda} = \max(0, d - d^* - \nu)$. The probability of default is therefore strictly positive only when the noisy fiscal space ($d^* + \nu - d$) is strictly negative, and null otherwise. Standard results on truncated normal distributions allow to compute the default probability conditional on the fiscal space only ($d^* - d$)—i.e. integrating out over all possible values of the noise ν. Formally: $\mathbb{P}(\mathcal{D}_{t+1} = 1 | \mathcal{D}_t = 0, d^* - d_t = u) = 1 - f(u)$, with $f(u) = \Phi(u/\sigma_\nu) + \exp(\alpha u + \alpha^2 \sigma_\nu^2 / 2)(1 - \Phi(u/\sigma_\nu + \sigma_\nu \alpha))$. The one-year default probabilities—displayed on the figure—are given by $1 - f(u)^4$.

where e_i denotes the i^{th} column of the identity matrix of dimension $n_m \times n_m$. The definition of Ω implies in particular that $\mathbb{E}(m_{t+1} | m_t) = \Omega' m_t$.

Parameters ν_π and ν_y capture specific effects of the sovereign default on inflation and GDP growth, respectively. In other words, our model accommodates the potential macroeconomic impact of a sovereign default, aligning with what empirical evidence suggests (see Subsection 4.3.1).

3.4.3. Budget surplus

The primary budget surplus is given by:

$$s_t = s^* + \underbrace{\beta \times d_{t-1}}_{\text{stabilization component}} + \underbrace{\eta_t,}_{\text{risk component}} \tag{16}$$

where the term βd_{t-1} (with $\beta > 0$) captures the government's desire to stabilize the debt (as in, e.g., Bohn, 1998; Ghosh et al., 2013), and where the risk component η_t includes an exogenous shock (ε_t), but can also depend on the macroeconomic regimes m_t:

$$\eta_t = \varepsilon_t + \mu_\eta'(m_t - \mathbb{E}_{t-1}(m_t)) = \varepsilon_t + \mu_\eta'(m_t - \Omega' m_{t-1}).$$

The term $\mu_\eta'(m_t - \mathbb{E}_{t-1}(m_t))$ allows to introduce conditional correlation between primary budget surpluses and macroeconomic innovations. Typically, setting μ_η equal to a multiple of μ_y drives a correlation between surpluses and output growth, which is consistent with empirical evidence (e.g., van den Noord, 2000).

3.5. Solving the model

At this point, all components of the model have been presented. To generate outputs for a specific set of parameters and a given issuance strategy, we need to solve the model. This involves (A) determining function $q(.)$, which characterizes the price of the perpetuity issued by the government (see Subsection 3.2.1) and (B) solving for the stochastic discount factor (SDF) for specific agents' preferences (see Subsection 3.3.1). In our application, we consider Epstein-Zin preferences with a unit elasticity of intertemporal substitution.[15,16]

Note that these two tasks, namely (A) and (B), are intertwined since the computation of the SDF (task B) depends on how the default probability is affected by the state z_t,

[15]Using a unit EIS facilitates the resolution, i.e., the computation of the stochastic discount factor (e.g., Piazzesi and Schneider, 2007; Seo and Wachter, 2018, among others).

[16]While the SDF could be directly parameterized (by defining function f^r and ν^r of eq. 8 in an ad-hoc parametric way), considering standard preferences is useful to discipline the model calibration and keep it parsimonious. Typically, by considering Epstein-Zin preferences, we simply need to specify two additional parameters: a coefficient of risk aversion and the elasticity of intertemporal substitution (EIS). See Appendix II for details regarding the derivation of the SDF in the context of Epstein-Zin preferences with unit EIS.

whose dynamics depend on the pricing of perpetuities (task A), that, in turn, depends on the SDF. Appendix III describes the numerical strategy we employ to address these nested fixed-point problems.

4. Application

Within the model, the government can adjust its issuance strategy using three key levers: average debt maturity (via the coupon decay rate), inflation indexation (through κ_π), and indexation to real GDP growth (through κ_y). That is, a debt strategy is defined by $(\chi, \kappa_\pi, \kappa_y)$. This strategy impacts bond prices, debt dynamics, and ultimately, the likelihood of default. In Subsection 4.2, we examine the performances associated with different issuance strategies in the context of stylized economies. In Subsection 4.3, we conduct the same type of exercise in an economy calibrated to U.S. data. Before turning to these exercises, we present the metrics that we will use to compare the performance of the issuance strategy.

4.1. Measuring the debt strategy performances

A variety of metrics have been devised in both academic circles and by practitioners to assess the effectiveness of issuance strategies (e.g., Missale, 1999, 2012). While academics have mainly focused on debt sustainability, moral hazard, and the ability of a debt structure to shield the government from unforeseen fiscal shocks (see Section 1), practitioners have concentrated on metrics concerning the cost and risk associated with debt charges (e.g., Bergstrom et al., 2002; Pick and Anthony, 2008; Bolder and Deeley, 2011; Balibek and Memis, 2012; Bernaschi et al., 2019).

This framework allows for the calculation of various cost and risk metrics related to interest expenditures (r_t) or debt (d_t). Notably, these calculations are based on analytical formulas derived from the model—once it is solved—rather than on Monte-Carlo simulations.[17] In the remaining of this paper, we consider the following set of performance measures:

[17] Bernaschi, Missale, and Vergni (2009) discuss potential issues arising from simulation-based frameworks.

- **Average debt-to-GDP ratio and average debt service**

  These metrics, denoted by $\mathbb{E}(d_t)$ and $\mathbb{E}(r_t)$, are expressed as fractions of GDP. Both criteria reflect the funding costs associated with the different strategies. Although these two metrics are tightly connected with each other, they are not equivalent due to nonlinearities involved in the debt-to-GDP ratio dynamics.

- **Debt volatility**

  Debt volatility is measured by $\sqrt{\mathbb{V}ar(d_t)}$ and $\sqrt{\mathbb{V}ar(\Delta d_t)}$. While the former captures the dispersion of debt-to-GDP values, the latter characterizes the volatility of changes in the indebtedness from one year to the other.

- **Upper tail of the debt-to-GDP distribution**

  The 95^{th} percentile of the debt-to-GDP distribution, denoted by $q_{95}(d_t)$, is used to characterize the right tail of the debt distribution.

- **Debt service volatility**

  The debt service volatility is measured by $\sqrt{\mathbb{V}ar(r_t)}$. Consistently with international debt accounting standards, debt service include debt indexation (to inflation and GDP).

- **Credit risk**

  Credit risk is measured by the average 10-year probability of default. It is formally given by $\mathbb{E}(\mathbb{P}(\mathcal{D}_{t+10}|\mathcal{D}_t = 0, z_t))$; we denote it by $\mathbb{P}(PD)$ in tables and figures.

- **Credit-risk costs**

  Credit risk costs are measured by the average 10-year credit spread, that is formally given by $\mathbb{E}(y_{t,10} - y^*_{t,10})$, where $y_{t,10}$ is the yield-to-maturity of a defaultable zero-coupon nominal bond of maturity 10 years, and where $y^*_{t,10}$ is the yield-to-maturity of an equivalent nondefaultable bond (featuring a recovery rate RR of 1). It is denoted by $\mathbb{E}(spd)$ in figures and tables.

Most of these measures do not represent purely cost or risk metrics. For example, the 95th percentile of the debt-to-GDP distribution of the average probability of default are influenced not only by the average cost associated with a particular strategy but also by the volatility of interest rates, inflation, GDP growth, the budget surplus and, more generally, by the dynamic covariances between these variables.

Our metrics are unconditional; they reflect an average situation. An alternative would be to consider the calculation of performance measures that are conditional on a specific situation—characterized by a given debt and debt service, say. For instance, instead of computing the average 10-year default probability $\mathbb{E}(\mathbb{P}(\mathcal{D}_{t+10}|\mathcal{D}_t = 0, z_t))$, we could consider the probability of default conditional on a given state z_t, i.e., $\mathbb{P}(\mathcal{D}_{t+10}|\mathcal{D}_t = 0, z_t)$. It may indeed be the case that some strategies are better to deal with some specific situations (in the medium run). While this type of analysis can be carried out by using the present framework, it is beyond the scope of the present paper.

4.2. Insights from stylized economies

Before presenting the results of our baseline calibration (in Subsection 4.3), this subsection highlight key insights in the context of two simplified economies that are almost identical, with the exception of the correlation between inflation and GDP growth.

4.2.1. Presentation of the two economies

The first economy is influenced by demand shocks, exhibiting a positive correlation between inflation and GDP growth. In contrast, the second economy is characterized by supply shocks, resulting in a negative correlation between inflation and GDP growth. The selection of these two economies is guided by the significant implications that the relative importance of demand and supply shocks has on the term structure of bond returns (e.g., Rudebusch and Swanson, 2012; Campbell et al., 2017; Bekaert et al., 2021).[18]

The parameterizations of these two economies are close (see Table 2). Both economies comprise three regimes representing low, medium, and high output growth, with identical transition probabilities. Inflation rates also have three distinct values (low, medium,

[18] For a discussion on the significance of the relative impact of demand and supply shocks on debt management, see also Missale (2012).

and high). However, in the demand-driven economy, high inflation and high output growth occur within the same regime, while this relationship is reversed in the supply-driven economy.

Table 2: Stylized models: parameterizations

Regime	μ_π		μ_y	Ω		
	D	S				
1	0.000	0.060	0.000	0.800	0.200	0.000
2	0.030	0.030	0.020	0.100	0.800	0.100
3	0.060	0.000	0.040	0.000	0.200	0.800

Notes: This table shows the parameterizations of the stylized demand/supply models. We also have $\alpha = 0.10$, $\beta = 0.10$, $\gamma = 10$, $\beta = 0.10$, $d^* = 1.00$, $s^* = -0.08$, $\sigma_\nu = 0.10$, $\nu_\pi = 0.00$, $RR = 0.50$.

4.2.2. Term structures of bond returns

To gain insight into bond prices in these economies, which will determine the government's funding costs, we compute the returns associated with different types of bonds (nominal, inflation-linked, and GDP-linked) and different maturities. To facilitate the comparison, we focus on the annualized expected nominal returns of these bonds (until maturity).[19] For a nominal bond, this measure coincides with the yield-to-maturity of the bond; for an inflation-linked bond, this is close to the sum of the bond real rate and the annualized expected inflation until maturity. At this point, we abstract from credit risk (considering, e.g., that $\alpha = 0$ in eq. 13); accordingly, bond prices depend only on the macroeconomic regimes m_t.

Figure 4 displays the resulting term structures of expected returns. The average term structures of expected returns for inflation-linked bonds (ILBs) and GDP-linked bonds (GDP-LBs) are relatively similar when comparing the two economies. This is because the payoffs of both ILBs and GDP-LBs are protected against inflation, which is the sole distinction between the two economies. The term structure of ILB returns is downward sloping across maturities, a characteristic commonly observed for the term structure of real rates in simple equilibrium models (see, e.g., Piazzesi and Schneider, 2007).[20] In

[19]See the caption of Figure 4 for computational details.

[20]This occurs because, in these equilibrium models (as is the case in the present stylized economies), real rates are typically positively correlated with output growth. Consequently, the prices of inflation-linked

contrast, the returns on nominal bonds vary significantly between the two models (indicated by blue lines): in the demand-driven economy, nominal bond returns are low and display a downward-sloping term structure due to negative inflation risk premiums. Conversely, in the supply-driven economy, positive inflation risk premiums lead to high nominal yields and an upward-sloping term structure. This disparity arises because, when demand shocks are prevalent, long-term nominal bonds tend to yield higher returns during unfavorable economic conditions (since inflation is typically low in periods of low growth). Consequently, investors are more inclined to purchase these bonds despite the low returns, particularly for long-term nominal bonds, which explains the negative slope of the term structure in the demand-driven economy.

Naturally, introducing credit risk (by setting $\alpha > 0$ in eq. 13) drives credit spreads between the returns examined above—free of credit risk—and government returns. Figure 4 shows for instance the change in nominal yields when α goes from 0 (blue solid line) to 0.1 (blue dashed line). It appears that average credit spreads are higher in the context of the supply-driven economy, which reflects the fact that default probabilities are lower in the context of the demand-driven economy if the government issues nominal perpetuities—funding costs associated with nominal securities are indeed low in demand-driven economies.[21]

The macroeconomic impact of a sovereign default, captured by ν_y and ν_π (see eq. 15), can also affect the performance of issuance strategies due to its influence on the dynamics of security pricing. When $\nu_y < 0$, a sovereign default leads to a drop in consumption, classifying it as a "disaster" in the sense of Barro (2006) or Gabaix (2012). In this situation, as discussed in Renne and Pallara (2024), an increase in debt has two opposing effects on government bond yields: the elevated probability of disaster tends to reduce risk-free rates, while the credit spread component of the government yield increases. In

bonds (ILBs) tend to rise during recessions, providing them with an insurance value that increases with the bond's maturity. As a result, investors are willing to purchase long-term ILBs even if they offer lower returns compared to short-term ILBs, leading to downward-sloping term structures (see Campbell, 1986).

[21]Credit spreads are influenced by the type of perpetuities issued by the government. Here, we consider the prices of nominal zero-coupon bond in an economy where the government issues nominal perpetuities with a given duration; the prices of the same type of nominal zero-coupon bonds would be different if the government was issuing other types of perpetuities (because credit risk would then be different).

certain cases, the first effect can dominate the second, meaning that increased debt can lead to a decrease in government yield, thereby reducing the usual snowball effect. This is illustrated by Figure 3 that shows, for our two economies, the relationship between the debt-to-GDP ratio and yields-to-maturity of two types of nominal perpetuities: the first is a counterfactual risk-free perpetuity (i.e., with a recovery rate $RR = 1$) and the second is the perpetuity issued by the government ($RR < 1$). When $\nu_y = 0$ (left-hand plots), risk-free returns are independent of debt levels and depend solely on the three macro regimes m_t, as represented by the three horizontal grey lines. Meanwhile, government rates increase with indebtedness due to the credit spread component. However, when a sovereign default leads to a 10% reduction in consumption ($\nu_y = -0.10$, right-hand plots), risk-free rates are negatively affected by d_t, which drives government rates down despite the rise in the credit spread component.

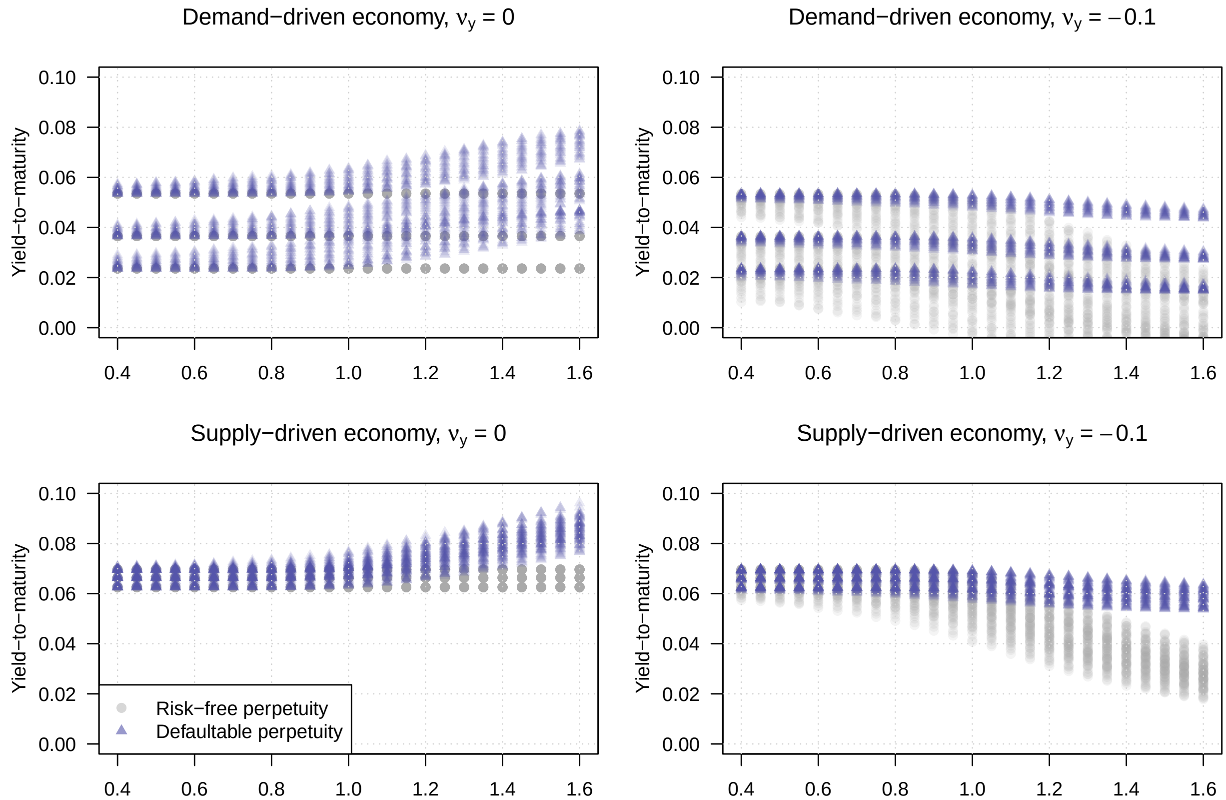
4.2.3. Debt-management performances

Table 3 shows the performances of issuance strategies that consist in issuing three basic types of perpetuities: a nominal one ($\kappa_\pi = 0$ and $\kappa_y = 0$), an inflation-linked one ($\kappa_\pi = 1$ and $\kappa_y = 0$), and a nominal-GDP-linked one ($\kappa_\pi = 1$ and $\kappa_y = 1$), in the context of the two stylized economies. We also consider two different values of χ: 0.2 (low duration, upper panel of the table) and 0.9 (high duration, lower panel).

The average debt service is tied to the average returns of issued perpetuities. Typically, in the context of the demand-driven economy, we expect lower average funding costs when nominal perpetuities are issued ($\kappa_\pi = 0$ and $\kappa_y = 0$) since nominal-bond yields are particularly low in this context (see Figure 4). Moreover, since the average term structure of nominal-bond yields is then downward-sloping, the higher χ—i.e., the long the perpetuity duration—the lower the debt service. More generally, we expect the respective locations of the yields shown in Figure 4 to play a determining role in the ranking of the average debt/GDP ratios associated with the different issuance strategies.

The previous reasoning is confirmed by the columns $\mathbb{E}(d)$ and $\mathbb{E}(r)$ of Table 3. For instance, the issuance of nominal perpetuities yields the cheapest (respectively most expensive) funding costs in the context of a demand-driven (supply-driven) economy.

Figure 3: Effect of ν_y on the relationship between debt and perpetuities' prices.

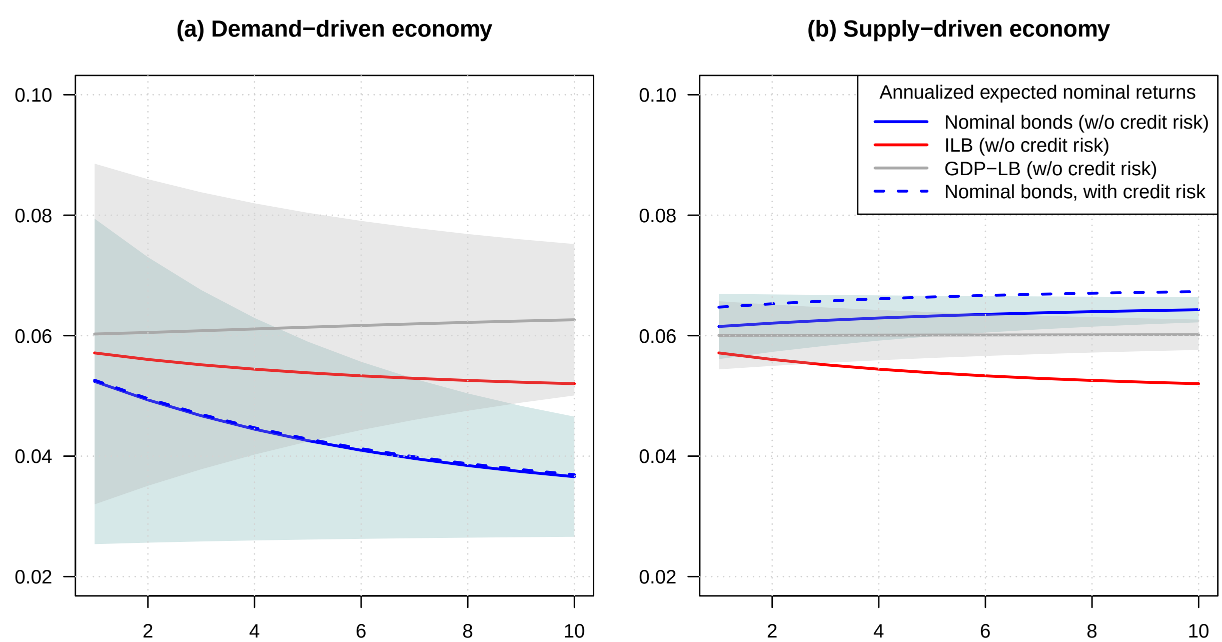


Note: This figure illustrates the relationship between indebtedness and the yield-to-maturity of perpetuities. We examine two types of perpetuities: a hypothetical risk-free one and the one issued by the government. The government issues nominal perpetuities, characterized by $(\chi, \kappa_\pi, \kappa_y) = (0.7, 0, 0)$. The specifications for the two models, demand-driven and supply-driven, are detailed in Table 2. The models underlying the left and right plots differ in the parameter ν_y, as indicated in the plot titles.

In line with the literature supporting the issuance of GDP-linked bonds for their stabilization properties, our findings indicate that issuing GDP-linked bonds results in less volatile debt, as shown in the columns $\sqrt{\mathbb{V}(d)}$ and $\sqrt{\mathbb{V}(\Delta d)}$ in Table 3. However, since investors demand higher return to hold these bonds (as their returns are procyclical), the associated average debt-to-GDP ratios turn out to be relatively high, implying poor overall performances. More specifically, issuing GDP-LBs is the worst strategy in term of sovereign credit risk in the context of a demand-driven economy, and it is dominated by the ILB-based strategy in the supply-driven economy—see column $\mathbb{E}(PD)$.

Contrary to GDP-LBs and ILBs, the performances of the nominal perpetuity strategy strongly depends on the nature of the shocks driving the economy. More precisely, while nominal bonds achieve the lowest levels of cost and risk when demand shocks prevail, it is the opposite when supply shocks prevail. The differences in the performances of nominal perpetuities—between the demand-driven and the supply-driven economies—are particularly marked when considering large debt duration ($\chi = 0.9$, lower panel). This is because, when the debt duration is low, the debt service associated with nominal perpetuities is tied to the short term rate, which is itself correlated to nominal GDP growth. When nominal perpetuities with longer durations are issued, the debt service correlate to a persistent moving average of previous long-term rates. The correlation of the debt service with the nominal GDP is then weaker, limiting the hedging capacity stemming from interest payments.

Figure 4: Average returns in the stylized economies (demand-driven and supply-driven)



Note: This figure shows the average expected nominal returns in the context of the demand-driven and supply-driven economies presented in Subsection 4.2. That is, consider a bond that provides a payoff of $\Pi_{t+h}^{\kappa_\pi} Y_{t+h}^{\kappa_y} / (\Pi_t^{\kappa_\pi} Y_t^{\kappa_y})$ (expressed in currency units) on date $t+h$; denoting its date-t price by $B_{t,h}$, the expected nominal payoff is given by $\mathbb{E}_t(\Pi_{t+h}^{\kappa_\pi} Y_{t+h}^{\kappa_y})/(B_{t,h}\Pi_t^{\kappa_\pi} Y_t^{\kappa_y})$. The shaded areas indicate the one-standard-deviation areas; they give a sense of the volatility of the returns; for readability, they are given for nominal and GDP-LBs only. The term structures represented by solid lines are computed in the absence of credit risk ($\alpha = 0$). The blue dashed line represents the term structure of government bond zero-coupon nominal yields in a context with credit risk ($\alpha > 0$), when the government issues perpetuities defined by $(\kappa_\pi, \kappa_y, \chi) = (0, 0, 0.7)$ (these therefore are nominal perpetuities).

Table 3: Performances of debt issuance strategies in stylized versions of the model, $\mu_\eta = 0 \times \mu_y$ and $\nu_y = 0$

	$\mathbb{E}(d)$	$\sqrt{\mathbb{V}(d)}$	$q_{95}(d)$	$\mathbb{E}(r)$	$\sqrt{\mathbb{V}(r)}$	$\sqrt{\mathbb{V}(\Delta d)}$	$\mathbb{E}(PD)$	$\mathbb{E}(spd)$
Coupon decay rate $\chi = 0.2$								
	Demand-driven economy ($\chi = 0.2$)							
Nominal	85.94	8.26	98.05	4.18	1.82	3.01	0.94	6.98
ILB	89.78	6.48	98.57	4.94	2.45	2.35	1.28	7.44
GDP-LB	94.58	5.55	99.97	5.65	3.14	2.38	2.30	11.59
	Supply-driven economy ($\chi = 0.2$)							
Nominal	97.18	7.49	107.40	6.08	0.77	2.39	3.58	17.18
ILB	89.79	6.50	98.59	5.04	1.42	2.35	1.28	7.46
GDP-LB	94.97	5.24	100.01	5.77	0.76	2.36	2.38	11.99
Coupon decay rate $\chi = 0.9$								
	Demand-driven economy ($\chi = 0.9$)							
Nominal	76.52	11.76	93.30	2.82	0.36	3.03	0.44	3.78
ILB	84.71	8.00	96.03	4.16	1.64	2.33	0.73	4.64
GDP-LB	94.38	5.94	100.95	5.68	3.21	2.38	2.31	11.56
	Supply-driven economy ($\chi = 0.9$)							
Nominal	100.96	8.18	112.55	6.63	0.65	2.28	5.38	25.31
ILB	85.25	8.48	98.14	4.36	1.95	2.33	0.90	5.68
GDP-LB	94.36	5.67	100.40	5.72	0.73	2.36	2.26	11.31

Notes: This table shows performance metrics associated with three different debt issuance strategies; each strategy consists in issuing a given type of perpetuities: a nominal perpetuity ($\kappa_\pi = 0$ and $\kappa_y = 0$), an inflation-indexed perpetuity nominal ($\kappa_\pi = 1$ and $\kappa_y = 0$), and a GDP-indexed perpetuity nominal ($\kappa_\pi = 1$ and $\kappa_y = 1$). We consider two different values of χ (the higher χ, the higher the average debt maturity). 'd' denotes the debt-to-GDP ratio. 'r' denotes the debt service, including debt indexation (in percent of GDP). '$\sqrt{\mathbb{V}(x)}$' corresponds to the standard deviation of variable x; 'PD' stands for '10-year probability of default' (expressed in percent); 'spd' stands for '10-year credit spread' (expressed in basis point), '$q_{95}(d)$' is the 95^{th} percentile of the debt-to-GDP distribution.

The previous results are obtained when their is no indexation of the surplus on real activity (i.e., $\mu_\eta = 0 \times \mu_y$ in eq. 16) and with no macroeconomic feedback of a sovereign default ($\nu_y = \nu_\pi = 0$ in eq. 15). Appendix V reports results obtained when $\mu_\eta = 1 \times \mu_y$ and $\nu_y = -10\%$. These additional results are qualitatively similar to those presented earlier; in particular, the performance rankings of the various strategies remain unchanged.

4.3. A calibrated economy

This section presents the results obtained in the context of an economy calibrated using U.S. data. Subsection 4.3.1 describes the calibration approach. Subsection 4.3.2 discusses the performances of different issuance strategies within this model.

4.3.1. Calibrating the model

The calibration involves the following parameters: those characterizing the dynamics of inflation and output growth (Π, μ_π, and μ_z, see eq. 15), the macroeconomic impact of a sovereign default (ν_y and ν_π), the coefficient of risk aversion (parameter γ of Epstein-Zin preferences, see Appendix II), the specification of the conditional probability of default (α and d^*, see eq. 13), the debt correction term in the fiscal reaction function (β in eq. 16).

Some of these parameters, namely ν_y, ν_π, and γ, are taken from the literature. We set ν_y to -5%, which is broadly in line with the average recessionary effect associated with sovereign defaults found by Mendoza and Yue (2012) and Reinhart and Rogoff (2011). The inflationary effect of a default (ν_π) is set to -2.1%, which is the average inflationary effect of a disaster used by Gabaix (2012). We set the coefficient of risk aversion to 10, a value also used by Bansal and Yaron (2004). Finally, in line with the estimates of van den Noord (2000), we consider that a one-percent increase in output improves the budget surplus by 0.5 percentage points, i.e., $\mu_\eta = 0.5 \times \mu_y$ (see eq. 16).

The estimation of Π, μ_π, and μ_z is the core step of the calibration process. With n_m macroeconomic regimes, there are $n_m(n_m + 1)$ parameters to estimate, amounting to 30 parameters for 5 regimes. The dynamics of macroeconomic variables has a critical importance to shape bond returns in the context of an equilibrium model such as ours.[22]

[22]According to Piazzesi and Schneider (2007), an equilibrium model is an asset pricing model characterized by (i) macroeconomic dynamics, particularly the dynamics of consumption, and (ii) agents' prefer-

Consequently, to ensure that the estimated macroeconomic dynamics yield realistic security prices, we incorporate yield curve data, alongside macroeconomic data, to guide the calibration of Π, μ_π, and μ_z. Besides, we employ an approach that places emphasis on both the macro-finance *fluctuations* and the *average* values of the yield curve. It is indeed critical to have a model that is consistent with plausible average bond returns. For example, if the parameterized model produces long-term real yields that are significantly lower than those observed in reality, it could lead to misleading conclusions when evaluating issuance strategies; this would indeed create the illusion of artificially low funding costs associated with issuing long-term ILBs.

In practice, denoting the vector of parameters to be estimated by Θ, we achieve our dual objective of capturing both macroeconomic fluctuations and long-term values by minimizing a loss function $L(\Theta) = -\log \mathcal{L}(\Theta) + d(\Theta)$, where (i) $\log \mathcal{L}(\Theta)$ is the log-likelihood function, which assesses the alignment of the parameterization with observed dynamics (the Hamilton, 1986, filter is used to compute this function), and (ii) $d(\Theta)$ is a measure of the distance between model-implied and targeted moments. Additional details regarding the estimation are provided in Appendix IV.

The resulting parameterization is displayed in Table 4. Table 5 documents the fit corresponding to the moment-matching part of the loss function. Figure 5 illustrates the time-series fit stemming from the Hamilton (1986) filter. Given the relatively small number of macroeconomic regimes ($n_m = 5$), the fit is necessarily imperfect. Using a small number of regimes is however consistent with our primary objective of capturing the main driving forces behind the joint fluctuations in macroeconomic variables and bond yields in a parsimonious and robust way. Figure 6 illustrates the model-implied term structures of average expected nominal returns for basic zero-coupon bonds, including nominal bonds, inflation-linked bonds (ILBs), and GDP-linked bonds (excluding credit risks). Notably,

ences. In this framework, the dynamic behavior of macroeconomic variables affects the risk premiums that agents require for holding the various assets examined.

the right-hand plot shows that, in the calibrated economy, investors would require significant excess returns—averaging over 200 basis points—to hold GDP-linked bonds.[23]

We next turn to the calibration of the remaining parameters. These includes β and s^*, that define the fiscal reaction function (eq. 16), as well as α, d^* and σ_ν, that are, respectively, the sensitivity of the probability of default to debt, the debt threshold, and the uncertainty regarding the latter (see eq. 13). These parameters are critical to convert a given state into probabilities of default and credit spreads. We set σ_ν to 10%, which is consistent with the order of magnitude of the uncertainty surrounding fiscal-limit estimates (see for instance Pallara and Renne, 2024). For the other parameters, we evaluate a range of values for each and seek the overall best fit for the following moments: the average 10-year credit spread (targeting 20 basis points) and the average and standard deviation of the debt-to-GDP ratio (with targets set at 80% and 15%, respectively).

Table 4: Model parameterization

Regime	μ_π	μ_y			Ω		
1	0.030	0.060	0.867	0.133	0.000	0.000	0.000
2	−0.016	−0.100	0.715	0.118	0.167	0.000	0.000
3	0.073	0.014	0.029	0.000	0.962	0.008	0.000
4	0.034	0.035	0.000	0.063	0.275	0.634	0.028
5	0.022	0.019	0.001	0.196	0.000	0.051	0.752

Notes: This table shows the model parameterization of the baseline model. We also have: $\alpha = 0.1$, $\beta = 0.20$, $\gamma = 10$, $\delta = 0.99$, $d^* = 1.10$, $s^* = -0.176$, $\nu_y = -0.050$, $\nu_\pi = -0.021$, $\mu_\eta = 0.5 \times \mu_y$, $RR = 0.50$.

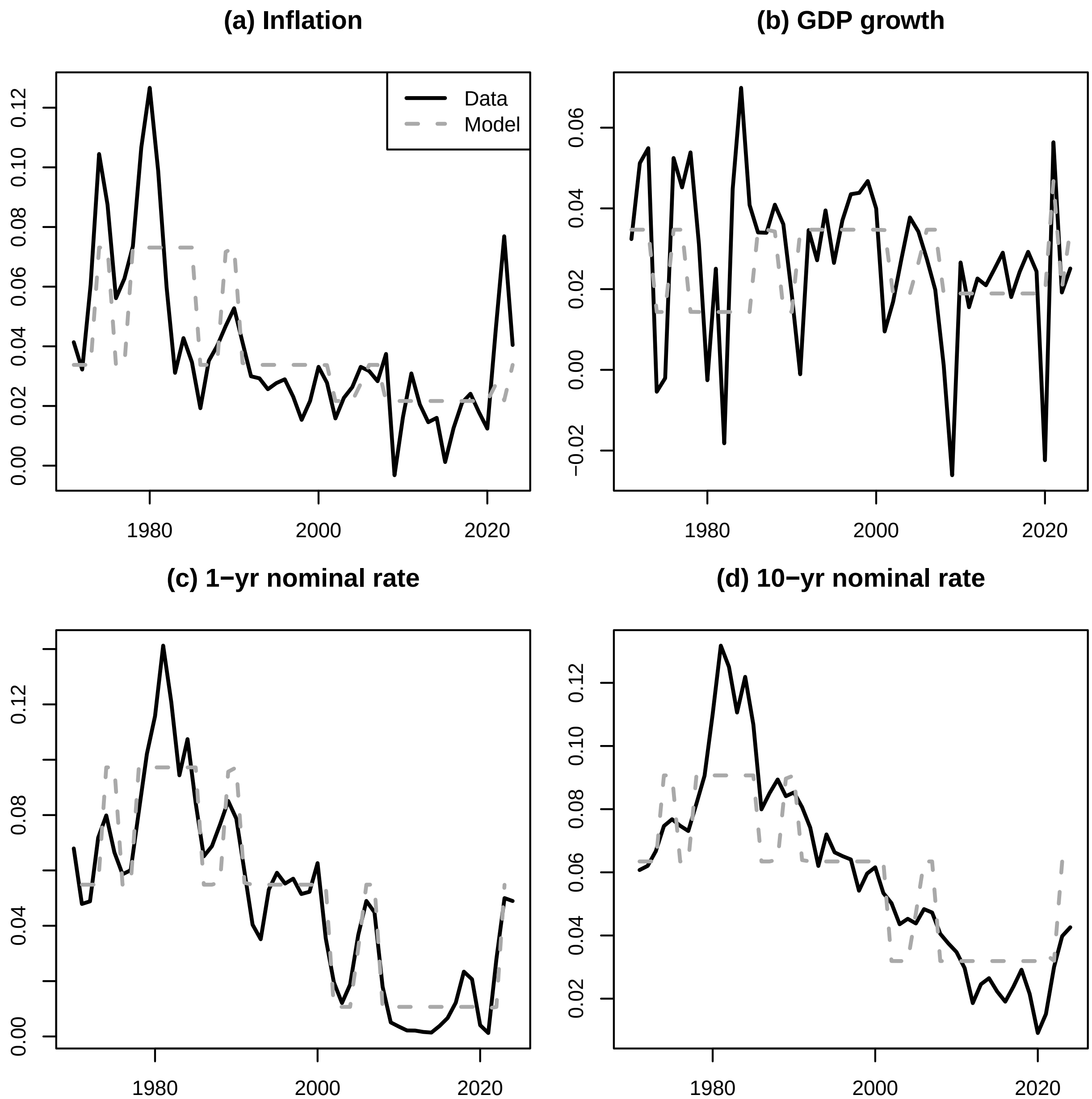
4.3.2. *Performance of debt issuance strategies in the calibrated model*

This section presents the performances of different issuance strategies implemented in the context of the calibrated model. The results are shown in Figure 7 and Table 6.

On each plot of Figure 7, a dot corresponds to a given strategy, characterized by an inflation indexation (κ_π), a real GDP indexation (κ_y), and an average maturity (captured by the coupon decay rate χ). We consider three values of χ: 0.1, 0.5, and 0.9. For a nominal yield-to-maturity of 6%, these values would correspond to durations of about 1, 2, and

[23]This estimate is at the upper end of the range of values available in the literature. For example, Mouabbi et al. (2024), Blanchard et al. (2016), Kamstra and Shiller (2009) obtain estimates of 40, 100 and 150 basis points, respectively.

Figure 5: Model fit



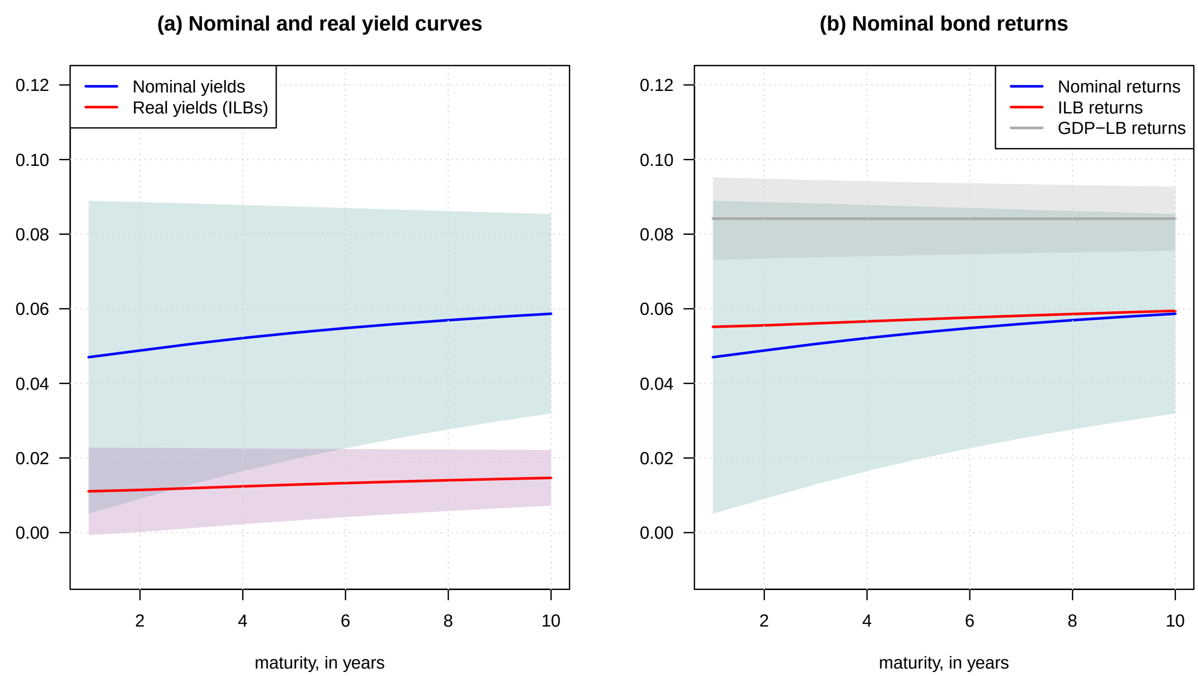
Note: This figure shows the fit resulting from the parameterization reported in Table 4, using the Hamilton (1986) filter. Yields are yields-to-maturity associated with zero-coupon bonds. Yields come from the Board of the Federal Reserve Bank, that publishes updates of yields computed by (Gürkaynak et al., 2007). The sample period is 1970-2023. See Subsection 4.3.1 for additional details on the estimation procedure. Inflation and output growth depend on m_t, a Markovian regime-switching chain (see Subsection 3.4.2). Nominal yields are derived from an equilibrium asset-pricing model; that is, their dynamics result from those of the macroeconomic variables and agents' preferences (see Subsection 3.3.1).

Table 5: Model-implied versus targeted moments

Moment	Model	Target
Avg. slope of nominal yield curve (1y-10y)	0.012	0.011
Avg. 10-year nominal yield	0.059	0.060
Avg. slope of real yield curve (2y-10y)	0.003	0.009
Avg. 10-year real yield	0.015	0.014
Avg. inflation	0.044	0.039
Avg. real GDP growth	0.029	0.027
Std dev. of 10-year nominal yield	0.027	0.030
Std dev. of 10-year real yield	0.007	0.013
Avg. breakeven	0.000	0.000

Notes: This table compares model-implied with targeted moments. The distance between these moments is part of the loss function that is minimized to estimate the components of μ_π, μ_y, and Ω. See Subsection 4.3.1 for more details.

Figure 6: Average nominal and real yield curves



Note: This figure displays unconditional returns of zero-coupon bonds resulting from the model whose parameterization is reported in Table 4. The left plot shows term structures of yields-to-maturity (real yields for ILBs, and nominal yields). The right plot displays the expected annualized nominal returns over the bond maturity; more precisely, consider a bond characterized by (κ_π, κ_y), this return is given by $-1/h(1/\mathcal{B}_{t,h}) \log \mathbb{E}_t((\Pi_{t+h}^{\kappa_\pi} Y_{t+h}^{\kappa_y})/(\Pi_t^{\kappa_\pi} Y_t^{\kappa_y}))$. (Indeed, the maturity nominal payoff of the bond is $\Pi_{t+h}^{\kappa_\pi} Y_{t+h}^{\kappa_y}$, and the price, on date t, is $\Pi_t^{\kappa_\pi} Y_t^{\kappa_y} \mathcal{B}_{t,h}$.) Shaded areas indicate one-standard-deviation areas. These returns do not include credit risk (i.e., $\alpha = 0$ in eq. 13).

6 years. We consider coefficients of indexations to inflation comprised between 0 and 1, but we limit the range of the coefficients to real GDP to $(0, 0.3)$ since larger coefficients of indexation lead to poor cost and risk performances due to the large risk premiums associated with GDP indexation. On the plots, the x-axis corresponds to a cost measure, namely the average debt-to-GDP; the y-axis corresponds to a risk measure: the standard deviation of the debt-to-GDP ratio for Panel (a), the 95th percentile of the debt-to-GDP ratio distribution for Panel (b), and the 10-year probability of default for Panel (c).

Table 6 reports the performances of a few basic strategies presenting the lowest/highest χ, κ_π, and κ_y. Moreover, the last three rows of the table show the results associated with those of the considered strategies that yield the best performances in terms of debt volatility ($\sqrt{\mathbb{V}(d)}$), 95th debt distribution percentile ($q_{95}(d)$), and 10-year probability of default ($\mathbb{E}(PD)$).

Table 6: Performances of debt issuance strategies in the calibrated model

$(\chi, \kappa_\pi, \kappa_y)$	$\mathbb{E}(d)$	$\sqrt{\mathbb{V}(d)}$	$q_{95}(d)$	$\mathbb{E}(r)$	$\sqrt{\mathbb{V}(r)}$	$\sqrt{\mathbb{V}(\Delta d)}$	$\mathbb{E}(PD)$	$\mathbb{E}(spd)$
(0.1, 0.0, 0.0)	80.25	13.07	99.31	3.97	3.73	7.52	0.20	9.08
(0.1, 0.0, 0.3)	82.91	11.09	99.39	4.75	3.22	6.98	0.19	5.94
(0.1, 1.0, 0.0)	82.22	11.54	99.26	4.70	3.36	6.96	0.18	4.68
(0.1, 1.0, 0.3)	86.22	9.20	100.30	5.69	2.80	6.18	0.22	4.30
(0.9, 0.0, 0.0)	84.44	10.77	101.49	4.97	1.91	8.51	0.41	23.03
(0.9, 0.0, 0.3)	86.71	9.47	102.00	5.57	1.85	7.42	0.39	17.14
(0.9, 1.0, 0.0)	84.40	10.05	99.38	5.04	2.68	7.21	0.22	7.66
(0.9, 1.0, 0.3)	93.01	9.44	106.30	7.41	2.99	6.10	0.73	16.87
(0.9, 0.6, 0.3)	88.55	8.58	101.10	6.10	2.23	6.46	0.33	10.87
(0.9, 0.4, 0.0)	83.77	9.96	98.61	4.90	2.09	7.49	0.25	13.49
(0.1, 1.0, 0.0)	82.22	11.54	99.26	4.70	3.36	6.96	0.18	4.68

Notes: This table shows performance metrics associated with different debt issuance strategies characterized by the issuance of perpetuities of different durations (captured by the coupon decay rate χ), a coefficient of indexation to inflation κ_π and a coefficient of indexation to real GDP κ_y. The model is the one whose parameterization is reported in Table 4. 'd' denotes the debt-to-GDP ratio. 'r' denotes the debt service, including debt indexation (in percent of GDP). '$\sqrt{\mathbb{V}(x)}$' corresponds to the standard deviation of variable x; 'PD' stands for '10-year probability of default' (expressed in percent); 'spd' stands for '10-year credit spread' (expressed in basis point), '$q_{95}(d)$' is the 95^{th} percentile of the debt-to-GDP distribution. The last three rows show the performances of the strategies implying the lowest $\sqrt{\mathbb{V}(d)}$, $q_{95}(d)$, and $\mathbb{E}(PD)$, respectively.

Let us start with two general observations. First, there are significant differences in the performance of various strategies. For example, the gap between the best and worst performers in terms of the average debt-to-GDP ratio is 15 percentage points, and 10 percentage points when considering the 95th percentile of the debt distribution (see Figure 7). Second, the ranking of strategies varies depending on the risk measure used. While strategies with greater indexation to real GDP result in a less dispersed debt distribution ($\sqrt{\mathbb{V}(d)}$), they perform relatively poorly for the other two risk measures, which are more relevant for assessing debt sustainability. This is because real GDP indexation incurs high funding costs, shifting the debt distribution to the right, similar to findings by Mouabbi, Renne, and Sahuc (2024).

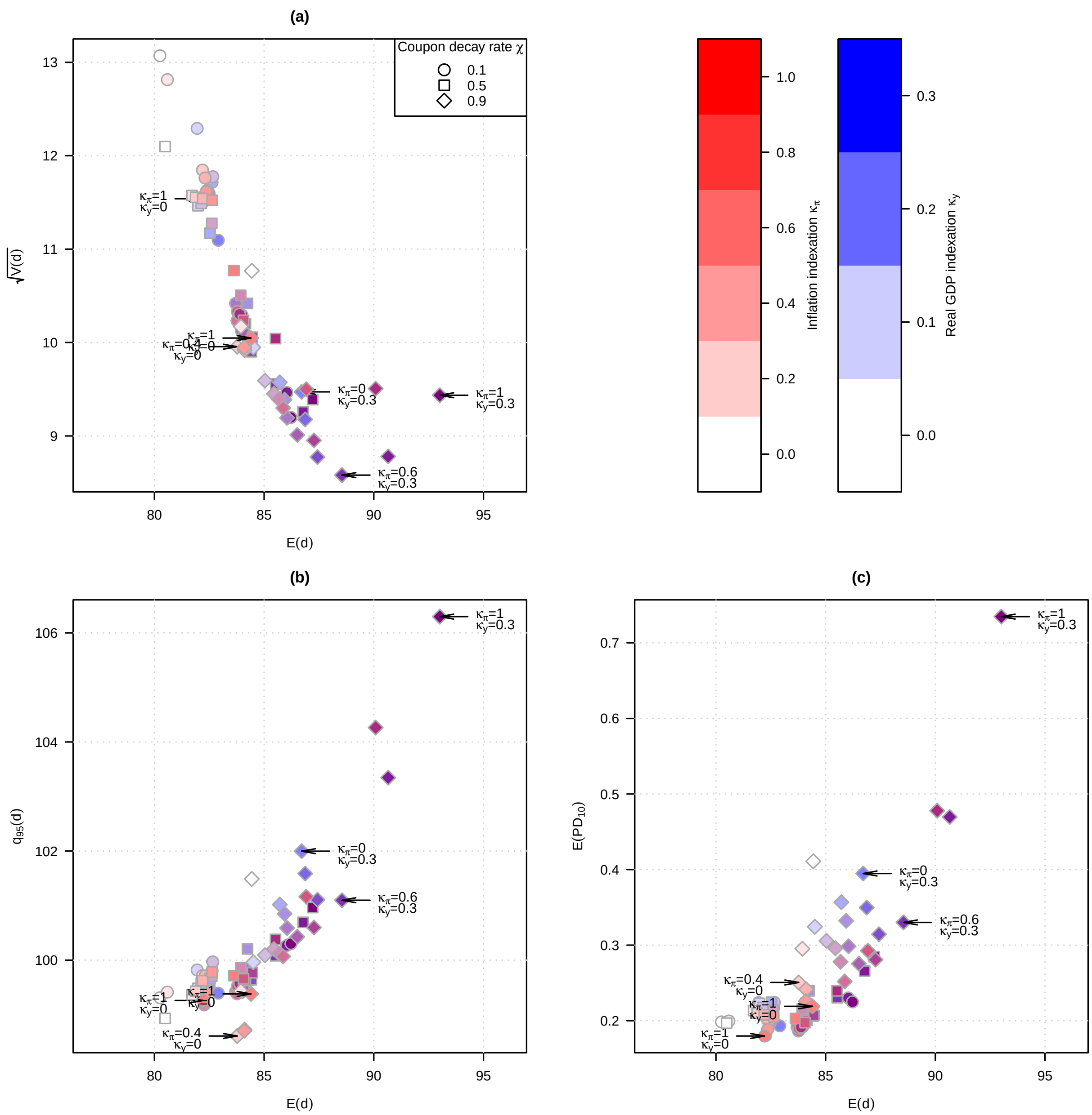
Our results suggest that the strategies minimizing the right tail of the debt-to-GDP distribution and the average probability of default do not involve any GDP indexation. By contrast, the lowest risk measures are associated with strategies characterized by significant inflation indexation—approximately 40% for $q_{95}(d)$ and 100% for $\mathbb{E}(PD)$. Consistent with the upward-sloping term structures of returns, the average debt level increases with debt maturity (determined by χ). While a large debt maturity is necessary to minimize $q_{95}(d)$, the minimum average probability of default occurs with the smallest maturity.

5. Concluding remarks

This paper presents an analytical framework to examine how government debt structure impacts debt sustainability, considering three key features of the debt portfolio: average maturity, inflation indexation, and GDP indexation. Our macro-finance framework addresses the interplay between debt accumulation and bond prices in a context entailing risk averse investors. The model handles exogenous macroeconomic shifts, making it suitable for real data application.

Our findings indicate that the assessment of the effectiveness of different debt issuance strategies depends significantly on the underlying model that produces the joint dynamics of security prices and macroeconomic variables such as inflation, real output, and budget surplus. We document, in particular, the strong influence that the correlation between inflation and GDP growth have on the cost and risk performances of nominal debt. A quantitative exercise based on calibrating the model with US data suggests that indexation to inflation—but not to real GDP—of the US debt portfolio helps to reduce the right tail of the distribution of debt-to-GDP ratios (compared with no indexation).

Figure 7: Cost and risk performances of debt issuance strategies



Note: This figure illustrates the cost and risk performances of different debt issuance strategies. On each plot, a dot corresponds to a given strategy. A strategy is defined by three characteristics: an inflation indexation (κ_π), a real GDP indexation (κ_y), and an average maturity (captured by the coupon decay rate χ). (See Subsection 3.2.1 for a description of the securities issued by the government.) The performances are measured by means of unconditional moments; the x-axis corresponds to a cost measure (average debt-to-GDP); the y-axis corresponds to a risk measure: standard deviation of the debt-to-GDP ratio for Panel (a), 95th percentile of the debt-to-GDP ratio distribution for Panel (b), and 10-year probability of default for Panel (c).

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— Appendix —

An Analytical Framework for Public Debt Management

Jean-Paul Renne

I. Proofs of Section 3

I.1. Proof of Proposition 1

This proof is an extension of Renne and Pallara (2024) to the case where the government issues indexed debt.

Let us denote by I_t the proceeds of date-t issuances and by X_t the resulting first payments (settled on date $t+1$), both I_t and X_t being expressed in units of the composite index. By definition of q_t, we have:

$$I_t = \sum_{j=1}^{\infty} \frac{\chi^{j-1} X_t}{(1+q_t)^j} = \frac{X_t}{1+q_t-\chi}.$$

Consider the date-t (residual) face value of the issuances that took place on date $t-h$. According to the concept of nominal valuation of debt securities (see International Monetary Fund, Bank for International Settlements and European Central Bank, 2015), this face value is computed as the sum of future associated payoffs $\chi^h X_{t-h}$, $\chi^{h+1} X_{t-h}, \dots$, discounted using the issuance yield-to-maturity q_{t-h}—that materialized on date $t-h$. This is equal to $\chi^h I_{t-h}$. As a consequence, and because current debt D_t is the sum of the (residual) face values of all past issuances (for $h \geq 0$), we obtain:

$$D_t = I_t + \chi I_{t-1} + \chi^2 I_{t-2} + \cdots = I_t + \chi D_{t-1}. \tag{I.1}$$

Using $X_t = (1+q_t-\chi)I_t = (1+q_t-\chi)(D_t - \chi D_{t-1})$, past debt issuances give rise to the following debt payments on date $t+1$:

$$\begin{aligned} & CF_{t+1} = X_t + \chi X_{t-1} + \chi^2 X_{t-2} + \dots \\ = \; & (1+q_t-\chi)(D_t-\chi D_{t-1}) + \\ & \chi(1+q_{t-1}-\chi)(D_{t-1}-\chi D_{t-2}) + \chi^2(1+q_{t-2}-\chi)(D_{t-2}-\chi D_{t-3}) + \dots \\ = \; & D_t - \chi D_t + q_t(D_t-\chi D_{t-1}) + \chi q_{t-1}(D_{t-1}-\chi D_{t-2}) + \chi^2 q_{t-2}(D_{t-2}-\chi D_{t-3}) + \dots \end{aligned} \tag{I.2}$$

Let us now take a cash-flow perspective. On date t, the sum of the issuance proceeds ($\Pi_t^{\kappa_\pi} Y_t^{\kappa_y} I_t$) and of the primary budget surplus (S_t) has to equate date-t payments associated with previous issuances ($\Pi_t^{\kappa_\pi} Y_t^{\kappa_y} CF_t$). That is:

$$\Pi_t^{\kappa_\pi} Y_t^{\kappa_y} I_t = \Pi_t^{\kappa_\pi} Y_t^{\kappa_y} CF_t - S_t. \tag{I.3}$$

Using Eq. (I.1), we get:

$$\Pi_{t+1}^{\kappa_\pi} Y_{t+1}^{\kappa_y} D_{t+1} = (I_{t+1} + \chi D_t)\Pi_{t+1}^{\kappa_\pi} Y_{t+1}^{\kappa_y} = \Pi_{T+1}^{\kappa_\pi} Y_{t+1}^{\kappa_y} CF_{t+1} - S_{t+1} + \chi D_t \Pi_{t+1}^{\kappa_\pi} Y_{t+1}^{\kappa_y} \tag{I.4}$$

Substituting for CF_{t+1} (Eq. I.2) into Eq. (I.4), we have:

$$\begin{aligned} & \Pi_{t+1}^{\kappa_\pi} Y_{t+1}^{\kappa_y} D_{t+1} \\ = \; & \Pi_{T+1}^{\kappa_\pi} Y_{t+1}^{\kappa_y} D_t - S_{t+1} + \\ & + \Pi_{T+1}^{\kappa_\pi} Y_{t+1}^{\kappa_y} \Big(\underbrace{q_t(D_t - \chi D_{t-1}) + \chi q_{t-1}(D_{t-1} - \chi D_{t-2}) + \chi^2 q_{t-2}(D_{t-2} - \chi D_{t-3}) + \dots}_{\text{interest payments on date } t+1 \,\equiv\, R_{t+1}} \Big), \end{aligned} \tag{I.5}$$

which gives

$$\left\{ \begin{array}{rcl} D_{t+1}^{(\$)} & = & D_t^{(\$)} \exp(\kappa_\pi \pi_{t+1} + \kappa_y \Delta y_{t+1}) - S_{t+1} + R_{t+1}^{(\$)} \\ R_{t+1}^{(\$)} & = & q_t \exp(\kappa_\pi \pi_{t+1} + \kappa_y \Delta y_{t+1}) \left[D_t^{(\$)} - \chi D_{t-1}^{(\$)} \exp(\kappa_\pi \pi_t + \kappa_y \Delta y_t) \right] + \\ & & \chi R_t^{(\$)} \exp(\kappa_\pi \pi_{t+1} + \kappa_y \Delta y_{t+1}), \end{array} \right. \tag{I.6}$$

where $D_{t+1}^{(\$)} = \Pi_{t+1}^{\kappa_\pi} Y_{t+1}^{\kappa_y} D_{t+1} / (\Pi_{t+1} Y_{t+1})$ and $R_{t+1}^{(\$)} = \Pi_{t+1}^{\kappa_\pi} Y_{t+1}^{\kappa_y} R_{t+1} / (\Pi_{t+1} Y_{t+1})$.

Using $d_t = D_t^{(\$)} / (\Pi_t Y_t)$, $\underline{r}_t = R_t^{(\$)} / (\Pi_t Y_t)$, and $r_t = \underline{r}_t + (\zeta_t - \underline{\zeta}_t) d_{t-1}$ leads to Proposition 1.

I.2. *Proof of Proposition 2*

Let us determine how $\mathcal{P}_t$ depends on q_{t+1}. On date $t+1$, the payoff of the perpetuity is:

$$\left\{ \begin{array}{ll} 1 + \chi \mathcal{P}_{t+1} & \text{if } \mathcal{D}_{t+1} = 0, \\ RR + \mathbb{E}\left(\sum_{h=2}^{\infty} \mathcal{M}_{t+1,t+h} \chi^{h-1} RR | \mathcal{D}_{t+1} = 1, z_{t+1} \right) & \text{if } \mathcal{D}_{t+1} = 1. \end{array} \right. \tag{I.7}$$

This payoff can also be written as follows:

$$\left\{ \begin{array}{ll} 1 + \chi \mathcal{P}_{t+1} & \text{if } \mathcal{D}_{t+1} = 0, \\ RR(1 + \chi \underline{\mathcal{P}}_{t+1}) & \text{if } \mathcal{D}_{t+1} = 1, \end{array} \right. \tag{I.8}$$

where

$$\underline{\mathcal{P}}_t = \mathbb{E}\left(\sum_{h=1}^{\infty} \mathcal{M}_{t,t+h}\chi^{h-1}\Big|\mathcal{D}_t = 1, z_t\right).$$

Using (I.8) with the SDF specification (10), and noting that $1 + \chi\mathcal{P}_{t+1} = \frac{1+q_{t+1}}{1+q_{t+1}-\chi}$, we have:

$$\begin{aligned}
\mathcal{P}_t &= \mathbb{E}\left(\mathcal{M}_{t,t+1}\left[\mathcal{D}_{t+1}RR\left(1+\chi\underline{\mathcal{P}}_{t+1}\right) + (1-\mathcal{D}_{t+1})(1+\chi\mathcal{P}_{t+1})\right]|\mathcal{D}_t = 0, z_t\right) \\
&= \mathbb{E}\left(e^{f(z_t,z_{t+1})}\left[\mathcal{D}_{t+1}\left(RRe^{\mu_v' z_{t+1}}(1+\chi\underline{\mathcal{P}}_{t+1}) - \frac{1+q_{t+1}}{1+q_{t+1}-\chi}\right) + \frac{1+q_{t+1}}{1+q_{t+1}-\chi}\right]\Bigg|\mathcal{D}_t = 0, z_t\right).
\end{aligned}$$

Eq. (11) is obtained by rearranging the terms of the previous equation, using $\mathcal{P}_t = 1/(1+q_t-\chi)$, and $\mathcal{P}_{t+1} = 1/(1+q_{t+1}-\chi)$.

I.3. Post-default prices of bonds and perpetuities

Proposition 4. *The post-default price of a zero-coupon bond of maturity h is given by:*

$$\underline{\mathcal{B}}_h(m_t) = \mathbf{1}'\left[\mathbf{D}(\exp(\mu_{f,1}))\Omega'\right]\left[\mathbf{D}(\exp(\mu_{f,0}+\mu_{f,1}))\Omega'\right]^{h-1}\left[\mathbf{D}(\exp(\mu_{f,0}))\right]m_t, \tag{I.9}$$

where $\mathbf{1} = [1,\dots,1]'$, where $\mathbf{D}(x)$ denotes a diagonal matrix whose diagonal entries are the entries of vector x, and where by abuse of notation, $\exp(x)$ is applied element-wise when x is a vector.

The post-default price of the perpetuity, that is

$$\underline{\mathcal{P}}_t = \mathbb{E}\left(\sum_{h=1}^{\infty} \mathcal{M}_{t,t+h}\chi^{h-1}\Big|\mathcal{D}_t = 1, z_t\right),$$

is given by

$$\underline{\mathcal{P}}(m_t) = \mathbf{1}'\left[\mathbf{D}(\exp(\mu_{f,1}))\Omega'\right]\left[Id - \chi\mathbf{D}(\exp(\mu_{f,0}+\mu_{f,1}))\Omega'\right]^{h-1}\left[\mathbf{D}(\exp(\mu_{f,0}))\right]m_t. \tag{I.10}$$

(Note that m_t is included in z_t.)

Proof. First, note that after default, debt dynamics are no longer important in determining bond prices. Consequently, the relevant pricing information, in z_t, is confined to m_t.

We have:

$$\begin{aligned}
\underline{\mathcal{B}}_h(z_t) &= \mathbb{E}(\mathcal{M}_{t,t+h}|\mathcal{D}_t = 0, z_t) \\
&= \mathbb{E}(\exp(\mu_{f,0}' m_t + (\mu_{f,0}+\mu_{f,1})'[m_{t+1}+\cdots+m_{t+h-1}] + \mu_{f,1}' m_{t+h})|z_t).
\end{aligned}$$

This leads to (I.9), using the results presented in Appendix A of Renne (2017).

Using the same calculation, we get:

$$\underline{\mathcal{P}}_t = \mathbf{1}' \sum_{h=1}^{\infty} \chi^{h-1} \left[\mathbf{D}(\exp(\mu_{f,1}))\Omega'\right] \left[\mathbf{D}(\exp(\mu_{f,0}+\mu_{f,1}))\Omega'\right]^{h-1} \left[\mathbf{D}(\exp(\mu_{f,0}))\right] m_t,$$

which gives (I.10). □

I.4. *Proof of Proposition 3*

We have:

$$\begin{aligned}
\mathcal{B}_{t,h} &= \mathbb{E}_t \left(\exp\left(f_{t+1} + \nu \mathcal{D}_{t+1}\right) \left[\mathcal{B}_{t+1,h-1}(1-\mathcal{D}_{t+1}) + RR\mathcal{D}_{t+1}\underline{\mathcal{B}}_{t+1,h-1}\right] |\mathcal{D}_t = 0\right) \\
&= \mathbb{E}_t \left(\mathcal{D}_{t+1} \exp\left(f_{t+1} + \nu\right) RR\underline{\mathcal{B}}_{t+1,h-1} |\mathcal{D}_t = 0\right) + \\
&\quad \mathbb{E}_t \left(\left(1-\mathcal{D}_{t+1}\right) \exp\left(f_{t+1}\right) \mathcal{B}_{t+1,h-1} |\mathcal{D}_t = 0\right) \\
&= \mathbb{E}_t \left(\exp(f_{t+1})\mathcal{B}_{t+1,h-1} + \mathcal{D}_{t+1} \exp(f_{t+1}) \left[e^{\nu} RR\underline{\mathcal{B}}_{t+1,h-1} - \mathcal{B}_{t+1,h-1}\right] |\mathcal{D}_t = 0\right),
\end{aligned}$$

which gives Eq. (12).

I.5. *Term structure of default probabilities*

Proposition 5. *Denote by $p_h(z_t)$ the horizon-h probability of default, that is:*

$$p_h(z_t) \equiv \mathbb{E}\big(\mathcal{D}_{t+h}|\mathcal{D}_t = 0, z_t\big).$$

These probabilities of default can be computed recursively, using:

$$p_h(z_t) = \mathbb{E}_t\big(\mathcal{D}_{t+1}[1 - p_{h-1}(z_{t+1})] + p_{h-1}(z_{t+1})|z_t, \mathcal{D}_t = 0\big), \tag{I.11}$$

starting from $p_0(z_t) \equiv 0$.

Proof. We have:

$$\begin{aligned}
p_h(z_t) &= \mathbb{E}_t\big(\mathbb{E}_{t+1}\big(\mathcal{D}_{t+h}|\mathcal{D}_t = 0\big)|\mathcal{D}_t = 0\big) = \mathbb{E}_t\big(\mathcal{D}_{t+1} + (1-\mathcal{D}_{t+1})p_{h-1}(z_{t+1})|\mathcal{D}_t = 0\big) \\
&= \mathbb{E}_t\big(\mathcal{D}_{t+1}[1 - p_{h-1}(z_{t+1})] + p_{h-1}(z_{t+1})|\mathcal{D}_t = 0\big),
\end{aligned}$$

which proves Eq. (I.11). □

II. Stochastic Discount Factor

This appendix explains how the SDF specification is obtained. Proposition 6 considers a situation where there is no feedback effect from a sovereign default on the macroeconomy (i.e., with $\nu_y = \nu_\pi = 0$ in eq. 15). Proposition 7, which considers the general case, uses Proposition 6 because the latter describes the post-default situation (that is taken into account in the expected continuation value).

Assumption 1. *The preferences of the representative agent are of the Epstein and Zin (1989) type, with a unit elasticity of intertemporal substitution (EIS). Specifically, the time-t log utility of a consumption stream (C_t) is recursively defined by*

$$u_t \quad = \quad \log U_t = (1-\delta)c_t + \frac{\delta}{1-\gamma}\log\left(\mathbb{E}_t \exp\left[(1-\gamma)u_{t+1}\right]\right), \tag{II.1}$$

where c_t denotes the logarithm of the agent's consumption level C_t, δ the pure time discount factor and γ is the risk aversion parameter.

Assumption 2. *The log growth rate of consumption (Δc_t) is given by $\mu_y' m_t$, where m_t is a selection vector following a homogeneous Markov-switching process characterized by a matrix of transition probabilities Ω (the sum of the coefficients of each row of Ω is equal to one).*

The following proposition gives the real SDF prevailing under the previous assumptions.

Proposition 6. *Under Assumptions 1 and 2, the SDF is given by*

$$\mathcal{M}^r_{t,t+1} \quad = \quad \exp\left[\underline{\mu^r_{f,0}}' m_t + \underline{\mu^r_{f,1}}' m_{t+1}\right],$$

where

$$\left\{\begin{array}{rcl} \underline{\mu^r_{f,0}} &=& \log(\delta)\mathbf{1} - \log\left\{\exp((1-\gamma)(\underline{\mu_u}+\mu_y))'\Omega'\right\}' \\ \underline{\mu^r_{f,1}} &=& (1-\gamma)\underline{\mu_u} - \gamma\mu_y, \end{array}\right. \tag{II.2}$$

where $\underline{\mu_u}$, which is such that $u_t = c_t + \underline{\mu_u}' m_t$, satisfies:

$$\underline{\mu_u} = \frac{\delta}{1-\gamma}\log\left\{\exp((1-\gamma)(\underline{\mu_u}+\mu_y))'\Omega'\right\}'. \tag{II.3}$$

Proof. When agent's preferences are as in eq. (II.1), the SDF is given by (e.g., Piazzesi and Schneider, 2007):

$$\mathcal{M}^r_{t,t+1} = \delta\left(\frac{C_{t+1}}{C_t}\right)^{-1}\frac{\exp[(1-\gamma)u_{t+1}]}{\mathbb{E}_t(\exp[(1-\gamma)u_{t+1}])}. \tag{II.4}$$

Let us first solve for the log-utility function. If u_t is given by $u_t = c_t + \underline{\mu_u}' m_t$, we have:

$$u_{t+1} = c_t + \Delta c_{t+1} + \underline{\mu_u}' m_{t+1} = c_t + (\underline{\mu_u} + \mu_y)' m_{t+1}.$$

Then, for a given state vector m_t, we have:

$$\text{eq. (II.1)} \quad \Leftrightarrow \quad c_t + \mu_u' m_t = (1-\delta)c_t + \delta c_t + \frac{\delta}{1-\gamma} \log\left\{ \exp((1-\gamma)(\underline{\mu_u} + \mu_y))'\Omega' \right\} m_t.$$

Therefore eq. (II.1) is satisfied for any state m_t iff (II.3) holds.

Using the exponential affine formulation of the utility in (II.4) leads to:

$$\begin{aligned}
\log \mathcal{M}^r_{t,t+1} &= \log \delta - \Delta c_{t+1} + (1-\gamma) u_{t+1} - \log \mathbb{E}_t(\exp[(1-\gamma)u_{t+1}]) \\
&= \log(\delta) - \mu_y' m_{t+1} + (1-\gamma)(\mu_y + \underline{\mu_u})' m_{t+1} \\
&\quad - \log \mathbb{E}_t (\exp \left\{ (1-\gamma) \left[(\mu_y + \underline{\mu_u})' m_{t+1} \right] \right\}) \\
&= \log(\delta) + \left[(1-\gamma)\underline{\mu_u} - \gamma \mu_y \right]' m_{t+1} - \log \mathbb{E}_t (\exp \left\{ (1-\gamma) \left[(\mu_y + \underline{\mu_u})' m_{t+1} \right] \right\}) \\
&= \log(\delta) + \left[(1-\gamma)\underline{\mu_u} - \gamma \mu_y \right]' m_{t+1} - \log \left\{ \exp((1-\gamma)(\underline{\mu_u} + \mu_y))'\Omega' \right\} m_t,
\end{aligned}$$

which gives the result. □

If $\nu_\pi \neq= 0$ or $\nu_y \neq= 0$ (see eq. 15), i.e., if the government default affects the macroeconomic variables, then Assumption 2 is not satisfied. However, the previous situation will be useful in the determination of the SDF in this more general situation (as it describes the post-default situation).

Let us replace Assumption 2 with the following one:

Assumption 3. *The log growth rate of consumption (Δc_t) is given by $\Delta c_t = \mu_y' m_t + \nu_y \Delta \mathcal{D}_t$, where m_t and $\hat{z}_t$ are selection vectors that are such that $m_t = M\hat{z}_t$ (i.e., $\hat{z}_t$ contains the information regarding the m_t chain). $\hat{z}_t$ follows a homogeneous Markov-switching process characterized by a matrix of transition probabilities Ω_z (the sum of the coefficients of each row of Ω_z is equal to one). This is therefore also the case of m_t; the matrix of transition probabilities of m_t is denoted by Ω (it derives from Ω_z). The dynamics of $\mathcal{D}_t$ depends on $\hat{z}_t$ through: $\mu_p' \hat{z}_t = \mathbb{P}(\mathcal{D}_{t+1} = 1 | \mathcal{D}_t = 0, \hat{z}_t)$.*

Proposition 7. *Under Assumptions 1 and 3, the SDF is given by*

$$\log \mathcal{M}^r_{t,t+1} = \hat{\mu}^{r'}_{f,0} \hat{z}_t + \hat{\mu}^{r'}_{f,1} \hat{z}_{t+1} + (\hat{\mu}^{r'}_{\nu} \hat{z}_{t+1}) \Delta \mathcal{D}_t, \tag{II.5}$$

where

$$\left\{\begin{array}{rcl}\hat{\mu}^r_{f,0} & = & \log(\delta)\mathbf{1} - \dfrac{1-\gamma}{\delta}\hat{\mu}'_u \\ \hat{\mu}^r_{f,1} & = & (1-\gamma)\hat{\mu}_u - \gamma\hat{\mu}_y \\ \hat{\mu}^r_v & = & (1-\gamma)(\underline{\hat{\mu}_u} - \hat{\mu}_u)'\hat{z}_{t+1} - \gamma\nu_c\mathbf{1}.\end{array}\right. \tag{II.6}$$

where $\hat{\mu}_y = M'\mu_y$*, and where* $\hat{\mu}_u$*, which is such that* $u_t = c_t + \hat{\mu}'_u\hat{z}_t$ *if* $\mathcal{D}_t = 0$*, satisfies (when* $\mathcal{D}_t = 0$*):*

$$\begin{aligned}\hat{\mu}'_u\hat{z}_t \quad = \quad & \frac{\delta}{1-\gamma}\log(\mathbb{E}_t\exp[(1-\gamma)(\hat{\mu}_u+\hat{\mu}_y)'\hat{z}_{t+1}] \times \\ & \{(\exp[(1-\gamma)(\nu_c + (\underline{\hat{\mu}_u} - \hat{\mu}_u)'\hat{z}_{t+1})] - 1)\Delta\mathcal{D}_{t+1} + 1\}),\end{aligned} \tag{II.7}$$

and with $\underline{\hat{\mu}_u} = M'\underline{\mu_u}$*, where* $\underline{\mu_u}$ *is obtained by applying Proposition 6.*

Proof. Let us first solve for the log-utility function. We posit that u_t is given by $u_t = c_t + \hat{\mu}'_u\hat{z}_t$ as long as $\mathcal{D}_t = 0$ and by $u_t = c_t + \underline{\hat{\mu}_u}'\hat{z}_t$ if $\mathcal{D}_t = 1$. We then have:

$$\begin{aligned}u_{t+1} &= (c_t + \Delta c_{t+1} + \hat{\mu}'_u\hat{z}_{t+1})(1-\mathcal{D}_{t+1}) + (c_t + \Delta c_{t+1} + \underline{\hat{\mu}_u}'\hat{z}_{t+1})\mathcal{D}_{t+1} \\ &= c_t + \Delta c_{t+1} + \hat{\mu}'_u\hat{z}_{t+1} + [(\underline{\hat{\mu}_u} - \hat{\mu}_u)'\hat{z}_{t+1}]\mathcal{D}_{t+1} \\ &= c_t + (\hat{\mu}_u + \hat{\mu}_y)'\hat{z}_{t+1} + \nu_c\Delta\mathcal{D}_{t+1} + [(\underline{\hat{\mu}_u} - \hat{\mu}_u)'\hat{z}_{t+1}]\mathcal{D}_{t+1} \\ &= c_t + (\hat{\mu}_u + \hat{\mu}_y)'\hat{z}_{t+1} + [\nu_c + (\underline{\hat{\mu}_u} - \hat{\mu}_u)'\hat{z}_{t+1}]\Delta\mathcal{D}_{t+1} \quad \text{if } \mathcal{D}_t = 0.\end{aligned}$$

Then, for a given state vector m_t, and assuming that $\mathcal{D}_t = 0$, we have:

$$\begin{aligned}\text{eq. (II.1)} \quad &\Leftrightarrow \quad c_t + \hat{\mu}'_u\hat{z}_t = (1-\delta)c_t + \frac{\delta}{1-\gamma}\log(\mathbb{E}_t\exp[(1-\gamma)u_{t+1}]) \\ &\Leftrightarrow \quad \hat{\mu}'_u\hat{z}_t = \frac{\delta}{1-\gamma}\log(\mathbb{E}_t\exp[(1-\gamma)\{(\hat{\mu}_u+\hat{\mu}_y)'\hat{z}_{t+1} + [\nu_c + (\underline{\hat{\mu}_u} - \hat{\mu}_u)'\hat{z}_{t+1}]\Delta\mathcal{D}_{t+1}\}]),\end{aligned}$$

which leads to (II.7).

Using the exponential affine formulation of the utility in (II.4) leads to:

$$\begin{aligned}\log\mathcal{M}^r_{t,t+1} &= \log\delta - \Delta c_{t+1} + (1-\gamma)u_{t+1} - \log\mathbb{E}_t(\exp[(1-\gamma)u_{t+1}]) \\ &= \log(\delta) - \hat{\mu}'_y\hat{z}_{t+1} - \nu_c\Delta\mathcal{D}_{t+1} + (1-\gamma)\{(\hat{\mu}_u+\hat{\mu}_y)'\hat{z}_{t+1} + [\nu_c + (\underline{\hat{\mu}_u} - \hat{\mu}_u)'\hat{z}_{t+1}]\Delta\mathcal{D}_{t+1}\} \\ &\quad -\log\mathbb{E}_t\left(\exp\left\{(1-\gamma)\{(\hat{\mu}_u+\hat{\mu}_y)'\hat{z}_{t+1} + [\nu_c + (\underline{\hat{\mu}_u} - \hat{\mu}_u)'\hat{z}_{t+1}]\Delta\mathcal{D}_{t+1}\}\right\}\right) \\ &= \log(\delta) + \left[(1-\gamma)\hat{\mu}_u - \gamma\hat{\mu}_y\right]'\hat{z}_{t+1} + [(1-\gamma)(\underline{\hat{\mu}_u} - \hat{\mu}_u)'\hat{z}_{t+1} - \gamma\nu_c]\Delta\mathcal{D}_{t+1} \\ &\quad - \underbrace{\log(\mathbb{E}_t\exp[(1-\gamma)\{(\hat{\mu}_u+\hat{\mu}_y)'\hat{z}_{t+1} + [\nu_c + (\underline{\hat{\mu}_u} - \hat{\mu}_u)'\hat{z}_{t+1}]\Delta\mathcal{D}_{t+1}\}])}_{=\frac{1-\gamma}{\delta}\hat{\mu}'_u\hat{z}_t},\end{aligned}$$

which proves (II.5). □

III. Numerical solution

This appendix describes the grid-based numerical strategy that we use to solve the model. Solving for the model amounts to jointly determining function $q(.)$ (see Proposition 2, and more precisely eq. 11) and the SDF specification (8) (see Proposition 7, and more precisely eq. II.7).

To start with, we need to determine the information that should be in the state vector. Given the dynamics (3)-(5), it comes that, for $(z_t, \mathcal{D}_t)$ to be Markovian, the state needs to include d_t, d_{t-1}, r_t, and m_t since these variables are necessary to predict d_{t+1} and r_{t+1} conditionally on the information collected until date t. (Note that π_t and Δy_t, and therefore ζ_t and $\underline{\zeta}_t$, depend on m_t, see eq. 15.) These variables define the state vector z_t: together, they turn out to be sufficient to describe the state of the economy described in Subsections 3.1 to 3.4, with agents' preferences are of the Epstein-Zin type (with $\Delta c_t = \Delta y_t$, where Δc_t is the consumption growth).[1]

In Subsection III.1, we consider the case where $\nu_y = 0$ and $\nu_\pi = 0$ (no direct macroeconomic impact of a government default, see eq. 15). Subsection III.2 deals with the general case; it makes use of the results presented in Subsection III.1 since the latter depicts the post-default situation, that has to be taken into account in the continuation value (even when $\mathcal{D}_t = 0$).

III.1. Case $\nu_y = 0$ and $\nu_\pi = 0$

In that case, Proposition 6 shows that the SDF specification depends on m_t only; solving for the SDF is fast if the dimension of m_t is limited (iterating on eq. II.3). It remains to solve for function $q(.)$; the arguments of this function are (d_t, d_{t-1}, r_t, m_t). We employ a grid-based numerical approach to approximate function $q(.)$. Specifically, we consider some sets of values for d_t, d_{t-1}, and r_t: n_d (identical) values for d_t and d_{t-1}, and n_r values for r_t. Consequently, and since m_t is a selection vector of dimension n_m, the state space is approximated by a finite number of states, i.e. $n_z := n_d^2 \times n_r \times n_m$. Accordingly, we employ a n_z-dimensional selection vector that we denote by $\hat{z}_t$. On each date, this selection vector points to the prevailing regime—among the n_z discretized states. We can therefore define vectors μ_d, $\mu_{d,-1}$ and μ_r such that, on any date t, d_t, d_{t-1} and r_t are respectively approximated with $\mu_d'\hat{z}_t$, $\mu_{d,-1}'\hat{z}_t$, and $\mu_r'\hat{z}_t$. Note that, although these vectors are of dimension $n_z \times 1$ (with $n_z > n_d$ and $n_z > n_r$), they respectively contain n_d, n_d, and n_r different entries.

[1]Assumptions 1 of Appendix II presents these preferences, for a unit elasticity of intertemporal substitution.

Recall that our objective is to solve for function $q(.)$. That is, we need to find a vector μ_q that is such that $q(z_t) \approx \mu_q' \hat{z}_t$, where $\hat{z}_t$ is the discretized version of z_t (i.e., on any date t, we have $d_t \approx \mu_d' \hat{z}_t$, $d_{t-1} \approx \mu_{d,-1}' \hat{z}_t$, and $r_t \approx \mu_r' \hat{z}_t$). The vector μ_q should approximately satisfy (11). We employ the following algorithm to obtain an approximation to μ_q:

(S1) We start from an initial guess for μ_q, that we denote $\mu_q^{(0)}$. This initial guess is obtained in the case where there is no credit risk, in which case $\mathcal{P}_t = \underline{\mathcal{P}}_t$, where $\underline{\mathcal{P}}_t$ is a function of m_t only (see Proposition 4).

(S2) At the i^{th} iteration, we approximate the right-hand side of (11) using $q_{t+1} \approx \mu_q^{(i-1)'} \hat{z}_{t+1}$, for all current (discretized) state z_t^*. This gives $\mu_q^{(i)}$.

The critical task, in the computation of the right-hand side of (11) concerns the evaluation of the expectation, that is:

$$\mathbb{E}_t \left(e^{f(z_t, z_{t+1}} \left[\mathcal{D}_{t+1} \left(RRe^{\nu^r(z_{t+1})}(1 + \chi \underline{\mathcal{P}}(z_{t+1})) - \frac{1 + q(z_{t+1})}{1 + q(z_{t+1}) - \chi} \right) + \frac{1 + q(z_{t+1})}{1 + q(z_{t+1}) - \chi} \right] \right),$$

where $\mathbb{E}_t(.) = \mathbb{E}(.|\mathcal{D}_t, z_t)$.

Let us detail on this computation. Exploiting the discrete nature of $\hat{z}_t$ and $\hat{z}_{t+1}$, we can evaluate the term within the expectation for each value of $\hat{z}_t$, $\hat{z}_{t+1}$, and $\mathcal{D}_{t+1}$. Hence, an approximation to the term within the expectation operator is of the form

$$\hat{z}_t' [M_1(\mu_q) \mathcal{D}_{t+1} + M_0(\mu_q)(1 - \mathcal{D}_{t+1})] \hat{z}_{t+1},$$

where $M_0(\mu_q)$ and $M_1(\mu_q)$ are two matrices of dimension $n_z \times n_z$ that depend in particular on the macroeconomic dynamics (15). With these notations, it comes that the conditional expectation appearing in (11) is of the form:

$$\mathbb{E}(\hat{z}_t' [M_1(\mu_q) \mathcal{D}_{t+1} + M_0(\mu_q)(1 - \mathcal{D}_{t+1})] \hat{z}_{t+1} | \mathcal{D}_t = 0, \hat{z}_t).$$

Using the notation $\mu_p' \hat{z}_{t+1} = \mathbb{P}(\mathcal{D}_{t+1} | \mathcal{D}_t = 0, \hat{z}_{t+1})$ (where μ_p stems from eq. 13), we obtain the following approximate expression of the conditional expectation:

$$\begin{aligned} & \mathbb{E}(\hat{z}_t' [M_1(\mu_q) \mathbf{D}(\mu_p) + M_0(\mu_q)(1 - \mathbf{D}(\mu_p))] \hat{z}_{t+1} | \mathcal{D}_t = 0, \hat{z}_t) \\ = \; & [M_1(\mu_q) \mathbf{D}(\mu_p) + M_0(\mu_q)(1 - \mathbf{D}(\mu_p))] \Omega_z' z_t, \end{aligned} \qquad \text{(III.1)}$$

where Ω_z denotes the transition matrix of $\hat{z}_t$ (the rows of Ω_z sum to one) and where $\mathbf{D}(x)$ denotes a diagonal matrix whose entries are the components of vector x.

Note that if n_d and n_r are large (to have sufficiently fine grids), then the dimension of Ω_z is very large, which can slow down the calculation. We handle this issue by noting that Ω_z is kept sparse if we consider discretized values of the budget surplus shock (ε_t in eq. 16), which is the only source of randomness of the model that has a real support.[2] Therefore, in order to speed up the computation, we consider a discretized set of draws for ε_t (n_ε values). That is, for each initial value of the approximate state vector $\hat{z}_t$, we consider $n_m \times n_\varepsilon$ potential outcomes (both for $\mathcal{D}_{t+1} = 0$ and $\mathcal{D}_{t+1} = 1$). This implies that each line of Ω_z contains $n_m \times n_\varepsilon$ non-zero entries. Hence, when evaluating matrices $M_0(\mu_q)$ and $M_1(\mu_q)$, we can focus on these nonzero entries.[3]

(S3) We iterate step 2 until convergence.

III.2. General case $\nu_y \neq 0$ and $\nu_\pi \neq 0$

In the general case, the previous problem is compounded with that of the SDF specification. Indeed, in that case, the SDF depends on $\mathcal{D}_t$ (see Proposition 7), whose dynamics depend on d_t (and therefore on z_t), which itself depends on the pricing of perpetuities, that, in turn, depends on the SDF. To address this issue, we expand Step S2 of the algorithm presented in Subsection III.1. Specifically, when considering a given $\mu_q^{(i)}$, we employ Proposition 7 to compute the SDF specification that would be consistent with this specific solution for the perpetuity price. That is, we start the i^{th} iteration of the algorithm of Subsection III.1 (within Step S2) by running another iterative algorithm (iterating eq. II.7) to solve for $\mu_u^{(i)}$ (say). The resulting SDF (eq. II.5) is then used to evaluate the conditional expectation of (11) along the lines described in Subsection III.1.

[2] Three sources of randomness underlie this expectation: (a) the budget surplus shock (ε_t in eq. 16), (b) changes in macroeconomic regimes (see Subsection 3.4.2), and (c) a potential default ($\Delta\mathcal{D}_t = 1$).

[3] For each of the $n_m \times n_\varepsilon$ outcomes per state $\hat{z}_t$, we need to determine the closest value of $\hat{z}_{t+1}$. For each of the resulting values of d_{t+1} and r_{t+1}, we look for the closest among the n_d and n_r values of the respective grids; this defines a unique value of $\hat{z}_{t+1}$.

IV. Estimation

This appendix provides details regarding the approach that is implemented to get estimates of μ_π, μ_y, and Ω, which characterize the dynamics of inflation and output growth (see Subsection 3.4.2). We gather these parameters in vector Θ.

As explained in Subsection 4.3.1, we achieve our dual objective of capturing both macroeconomic fluctuations and long-term values by minimizing a loss function $L(\Theta) = -\log \mathcal{L}(\Theta) + d(\Theta)$, where (i) $\log \mathcal{L}(\Theta)$ is the log-likelihood function, which assesses the alignment of the parameterization with observed dynamics (the Hamilton, 1986, filter is used to compute this function), and (ii) $d(\Theta)$ is a measure of the distance between model-implied and targeted moments. Here are some additional details regarding the computation of these two components:

(i) Since the model described in Subsection 3.4.2 is a Markov-switching model, the computation of the log-likelihood component involves the Hamilton (1986) filter. The state-space model involves four time series: inflation, real GDP growth, the one-year nominal rate and the ten-year nominal rate. The sample covers the period 1970-2023.[4,5]

(ii) As regards the distance function, the moments to match are the following: the average 10-year zero-coupon yields, the 1y-10y slope of the nominal yield curve, the 2y-10y slope of the real yield curves, the standard deviation of the 10-year real rate, the average inflation rate and the average output growth rate. We also include the average 10-year inflation risk premium in the set of moments to match; this risk premium is the difference between the average 10-year breakeven inflation rate—i.e., the difference between the 10-year nominal and real rates—and the average inflation rate. Except for the real rates, for which data start in 1999, the targets are the empirical moments calculated on the sample covering the period 1970-2023. Given the lack of consensus in the literature regarding the average value of the inflation risk premium, we set the associated target to zero, which is close to what Breach, D'Amico, and Orphanides (2020) find for the last 25 years.

To expedite the optimization of the loss function, we calculate the log-likelihood and the model-implied moments under the assumption that $\alpha = 0$ (absence of credit risk). This sim-

[4]While the first two series are extracted from the FRED database (using the mnemonics CPIAUCSL for the price index, and GDPC1 for the real GDP), nominal zero-coupon yields are based on the database initially constructed by Gürkaynak, Sack, and Wright (2007) and maintained by the Federal Reserve Bank.

[5]The standard errors of the measurement errors are as follows: 1% for the two macroeconomic variables, and 80 basis points for the nominal yields.

plification significantly streamlines the analysis, as bond returns then depend solely on the m_t Markov chain, and not on d_t and r_t. Consequently, we can proceed without numerically solving for the $q(.)$ function, see AppendixIII. (Note however that we do need to solve for the Stochastic Discount Factor (SDF) using Proposition6.) This approach implies that we are neglecting credit spreads in the model-implied yields. Nevertheless, this omission should not have strong implications since, over the past 20 years, U.S. CDS premiums have averaged less than 20 basis points, which is comparable to our fitting errors (see Figure 5).

V. Additional results

Table E.7: Performances of debt issuance strategies in stylized versions of the model, $\mu_\eta = 1 \times \mu_y$ and $\nu_y = 0$

	$\mathbb{E}(d)$	$\sqrt{\mathbb{V}(d)}$	$q_{95}(d)$	$\mathbb{E}(r)$	$\sqrt{\mathbb{V}(r)}$	$\sqrt{\mathbb{V}(\Delta d)}$	$\mathbb{E}(PD)$	$\mathbb{E}(spd)$
Coupon decay rate $\chi = 0.2$								
	Demand-driven economy ($\chi = 0.2$)							
Nominal	85.62	8.64	98.33	4.15	1.76	3.43	0.95	7.43
ILB	90.60	7.37	101.01	4.95	2.37	3.02	1.60	10.44
GDP-LB	94.69	5.40	99.96	5.64	3.12	2.38	2.31	12.09
	Supply-driven economy ($\chi = 0.2$)							
Nominal	97.21	7.58	107.66	6.09	0.82	2.37	3.62	18.14
ILB	90.64	7.33	100.99	5.12	1.52	3.02	1.60	10.46
GDP-LB	94.97	5.18	99.98	5.78	0.78	2.35	2.37	12.38
Coupon decay rate $\chi = 0.9$								
	Demand-driven economy ($\chi = 0.9$)							
Nominal	75.54	11.51	92.39	2.66	0.36	3.43	0.37	3.36
ILB	84.40	8.48	96.86	4.04	1.60	2.88	0.77	5.52
GDP-LB	94.66	5.62	100.41	5.65	3.17	2.47	2.34	12.27
	Supply-driven economy ($\chi = 0.9$)							
Nominal	102.14	8.97	115.28	6.80	0.85	2.29	6.10	29.99
ILB	85.97	8.94	99.49	4.45	2.03	2.87	1.06	7.53
GDP-LB	94.65	5.31	99.99	5.74	0.75	2.50	2.28	12.00

Notes: This table shows performance metrics associated with three different debt issuance strategies; each strategy consists in issuing a given type of perpetuities: a nominal perpetuity ($\kappa_\pi = 0$ and $\kappa_y = 0$), an inflation-indexed perpetuity nominal ($\kappa_\pi = 1$ and $\kappa_y = 0$), and a GDP-indexed perpetuity nominal ($\kappa_\pi = 1$ and $\kappa_y = 1$). We consider two different values of χ (the higher χ, the higher the average debt maturity). 'd' denotes the debt-to-GDP ratio. 'r' denotes the debt service, including debt indexation (in percent of GDP). '$\sqrt{\mathbb{V}(x)}$' corresponds to the standard deviation of variable x; 'PD' stands for '10-year probability of default' (expressed in percent); 'spd' stands for '10-year credit spread' (expressed in basis point), '$q_{95}(d)$' is the 95^{th} percentile of the debt-to-GDP distribution.

Table E.8: Performances of debt issuance strategies in stylized versions of the model, $\mu_\eta = 0 \times \mu_y$ and $\nu_y = 0.1$

	$\mathbb{E}(d)$	$\sqrt{\mathbb{V}(d)}$	$q_{95}(d)$	$\mathbb{E}(r)$	$\sqrt{\mathbb{V}(r)}$	$\sqrt{\mathbb{V}(\Delta d)}$	$\mathbb{E}(PD)$	$\mathbb{E}(spd)$
Coupon decay rate $\chi = 0.2$								
	Demand-driven economy ($\chi = 0.2$)							
Nominal	85.35	7.82	96.52	4.06	1.88	3.01	0.79	15.49
ILB	88.64	5.82	95.59	4.79	2.39	2.38	0.98	15.32
GDP-LB	91.66	6.23	99.34	5.21	3.20	2.43	1.64	19.37
	Supply-driven economy ($\chi = 0.2$)							
Nominal	93.82	6.01	101.02	5.60	0.52	2.33	2.15	26.91
ILB	88.70	5.92	96.07	4.88	1.37	2.38	1.00	15.60
GDP-LB	91.48	6.08	99.15	5.30	0.70	2.41	1.58	18.91
Coupon decay rate $\chi = 0.9$								
	Demand-driven economy ($\chi = 0.9$)							
Nominal	76.37	11.68	92.95	2.80	0.34	3.03	0.42	9.53
ILB	84.25	7.80	94.88	4.11	1.63	2.33	0.65	10.96
GDP-LB	92.72	6.16	99.71	5.37	3.16	2.37	1.88	22.50
	Supply-driven economy ($\chi = 0.9$)							
Nominal	95.95	6.86	104.81	5.92	0.41	2.30	3.00	35.75
ILB	84.03	7.82	94.86	4.19	1.87	2.33	0.64	10.81
GDP-LB	93.93	5.59	99.81	5.58	0.67	2.34	2.12	25.21

Notes: This table shows performance metrics associated with three different debt issuance strategies; each strategy consists in issuing a given type of perpetuities: a nominal perpetuity ($\kappa_\pi = 0$ and $\kappa_y = 0$), an inflation-indexed perpetuity nominal ($\kappa_\pi = 1$ and $\kappa_y = 0$), and a GDP-indexed perpetuity nominal ($\kappa_\pi = 1$ and $\kappa_y = 1$). We consider two different values of χ (the higher χ, the higher the average debt maturity). 'd' denotes the debt-to-GDP ratio. 'r' denotes the debt service, including debt indexation (in percent of GDP). '$\sqrt{\mathbb{V}(x)}$' corresponds to the standard deviation of variable x; 'PD' stands for '10-year probability of default' (expressed in percent); 'spd' stands for '10-year credit spread' (expressed in basis point), '$q_{95}(d)$' is the 95^{th} percentile of the debt-to-GDP distribution.