—Supplemental Appendix—

Time-varying risk aversion and inflation-consumption correlation in an equilibrium term structure model

Tilman BLETZINGER, Wofgang LEMKE, and Jean-Paul RENNE

I Proofs

I.1 Proof of Proposition 1

When agent's preferences are as in eq. (24), the s.d.f. is given by (e.g., Piazzesi and Schneider, 2007):

$$\mathscr{M}_{t,t+1} = \delta \left(\frac{C_{t+1}}{C_t}\right)^{-1} \frac{\exp[(1-\gamma_t)u_{t+1}]}{\mathbb{E}_t(\exp[(1-\gamma_t)u_{t+1}])}. \tag{I.1}$$

Let us show that, under Assumptions A.1 and A.2, the utility is exponential affine.

Proposition 8. *Under Assumptions A.1 and A.2, the utility* $U_t = C_t \exp(\mu_{u,0,t-1} + \mu'_{u,1}X_t)$*, or*

$$u_t = c_t + \mu_{u,0,t-1} + \mu'_{u,1}X_t \tag{I.2}$$

satisfies eq. (24) for any X_t *iff* $\mu_{u,1}$ *and* $\mu_{u,0,t}$ *are given by*

$$\mu_{u,1} = \delta(I_{n_X} - \delta\Phi')^{-1}\Phi'\mu_{c,1}, \tag{I.3}$$

and

$$\mu_{u,0,t} = -\mu_{c,0} + \frac{1}{\delta}\left\{\mu_{u,0,t-1} - \frac{\delta}{1-\gamma_t}\psi_0\left[(1-\gamma_t)(\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' + I_{n_X})\mu_{c,1}\right]\right\}. \tag{I.4}$$

Proof. If u_t is given by (I.2), we have:

$$u_{t+1} = c_t + \Delta c_{t+1} + \mu_{u,0,t} + \mu'_{u,1}X_{t+1} = c_t + \mu_{u,0,t} + \mu_{c,0} + (\mu_{u,1} + \mu_{c,1})'X_{t+1}.$$

Then, for a given state vector X_t, we have:

$$\begin{aligned}\text{eq. (24)} \quad \Leftrightarrow \quad & c_t + \mu_{u,0,t-1} + \mu'_{u,1}X_t \\ & = (1-\delta)c_t + \delta\mu_{u,0,t} + \delta\mu_{c,0} + \delta c_t + \\ & \frac{\delta}{1-\gamma_t}\left\{\psi_0\left[(1-\gamma_t)(\mu_{u,1}+\mu_{c,1})\right] + \psi_1\left[(1-\gamma_t)(\mu_{u,1}+\mu_{c,1})\right]'X_t\right\}\end{aligned}$$

$$
\begin{aligned}
\Leftrightarrow \quad & \mu_{u,0,t-1} - \delta \mu_{u,0,t} + \mu_{u,1}' X_t \\
&= \delta \mu_{c,0} + \frac{\delta}{1-\gamma_t} \left\{ \psi_0 \left[(1-\gamma_t)(\mu_{u,1} + \mu_{c,1}) \right] + \psi_1 \left[(1-\gamma_t)(\mu_{u,1} + \mu_{c,1}) \right]' X_t \right\}.
\end{aligned}
$$

Therefore eq. (24) is satisfied for any X_t iff the following two conditions are satisfied:

$$
\left\{
\begin{array}{l}
\mu_{u,0,t-1} - \delta \mu_{u,0,t} = \delta \mu_{c,0} + \frac{\delta}{1-\gamma_t} \psi_0 \left[(1-\gamma_t)(\mu_{u,1} + \mu_{c,1}) \right] \\
\mu_{u,1} = \frac{\delta}{1-\gamma_t} \psi_1 \left[(1-\gamma_t)(\mu_{u,1} + \mu_{c,1}) \right],
\end{array}
\right.
$$

which leads to the result, using that $\psi_1 \left[(1-\gamma_t)(\mu_{u,1} + \mu_{c,1}) \right] = (1-\gamma_t)\Phi'(\mu_{u,1} + \mu_{c,1})$ under Assumption A.2. □

Using the exponential affine formulation of the utility in (I.1) leads to:

$$
\begin{aligned}
\log \mathscr{M}_{t,t+1} &= \log \delta - \Delta c_{t+1} + (1-\gamma_t) u_{t+1} - \log \mathbb{E}_t(\exp[(1-\gamma_t) u_{t+1}]) \\
&= \log(\delta) - \mu_{c,0} - \mu_{c,1}' X_{t+1} + (1-\gamma_t)(\mu_{c,1} + \mu_{u,1})' X_{t+1} \\
&\quad - \log \mathbb{E}_t(\exp\left\{ (1-\gamma_t) \left[(\mu_{c,1} + \mu_{u,1})' X_{t+1} \right] \right\}) \\
&= \log(\delta) - \mu_{c,0} + \left[(1-\gamma_t)\mu_{u,1} - \gamma_t \mu_{c,1} \right]' X_{t+1} \\
&\quad - \log \mathbb{E}_t(\exp\left\{ (1-\gamma_t) \left[(\mu_{c,1} + \mu_{u,1})' X_{t+1} \right] \right\}) \\
&= \log(\delta) - \mu_{c,0} - \psi_0 \left[(1-\gamma_t)(\mu_{c,1} + \mu_{u,1}) \right] + \\
&\quad \left[(1-\gamma_t)\mu_{u,1} - \gamma_t \mu_{c,1} \right]' X_{t+1} - \psi_1 \left[(1-\gamma_t)(\mu_{c,1} + \mu_{u,1}) \right]' X_t.
\end{aligned}
$$

Hence, the stochastic discount factor between dates t and $t+1$ is given by:

$$
\mathscr{M}_{t,t+1} = \exp[-(\eta_{0,t}^* + \eta_1^{*\prime} X_t) + \lambda_t' X_{t+1} - \psi(\lambda_t, X_t)], \tag{I.5}
$$

with

$$
\left\{
\begin{array}{lcl}
\lambda_t &=& \left\{ (1-\gamma_t)\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' - \gamma_t I_{n_X} \right\} \mu_{c,1} \\
\eta_{0,t}^* &=& -\log(\delta) + \mu_{c,0} + \psi_0 \left[\lambda_t + \mu_{c,1} \right] - \psi_0(\lambda_t) \\
\eta_1^* &=& \Phi' \mu_{c,1}.
\end{array}
\right. \tag{I.6}
$$

Using $\gamma_t = \mu_{\gamma,0} + \mu_{\gamma,0}' X_t$ (Assumption A.2), we obtain:

$$
\begin{aligned}
\lambda_t &= \left[(1-\gamma_t)\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' - \gamma_t I_{n_X} \right] \mu_{c,1} \\
&= \left[\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' - \gamma_t(\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' + I_{n_X}) \right] \mu_{c,1} \\
&= \underbrace{\left[\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' - \mu_{\gamma,0}(\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' + I_{n_X}) \right] \mu_{c,1}}_{=\lambda_0} + \\
&\quad \underbrace{\left[-(\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' + I_{n_X})\mu_{c,1}\mu_{\gamma,1}' \right]}_{=\lambda_1'} X_t.
\end{aligned} \tag{I.7}
$$

Now, let's compute $\eta_{0,t}^*$. We have (using eq. I.6):

$$
\eta_{0,t}^* = -\log(\delta) + \mu_{c,0} + \psi_0(\lambda_t + \mu_{c,1}) - \psi_0(\lambda_t)
$$

$$= -\log(\delta) + \mu_{c,0} + \frac{1}{2}\mu_{c,1}'\Sigma\Sigma'\mu_{c,1} + \mu_{c,1}'\Sigma\Sigma'\lambda_t,$$

which proves Proposition 1.

I.2 Proof of Proposition 3

The conditional risk-neutral Laplace transform $\mathbb{E}_t^{\mathbb{Q}}(\exp(u'X_{t+1}))$ is equal to:

$$\begin{aligned}
& \mathbb{E}_t\left(\exp\left[u'X_{t+1}\right]\frac{\mathscr{M}_{t,t+1}}{\mathbb{E}_t(\mathscr{M}_{t,t+1})}\right) = \mathbb{E}_t\left(\exp\left[u'X_{t+1} + \lambda_t'\Sigma\varepsilon_{t+1} - \frac{1}{2}\lambda_t'\Sigma\Sigma'\lambda_t\right]\right) \\
= & \mathbb{E}_t\left(\exp\left[u'\Phi X_t + (u+\lambda_t)'\Sigma\varepsilon_{t+1} - \frac{1}{2}\lambda_t'\Sigma\Sigma'\lambda_t\right]\right) \\
= & \exp\left[u'\Phi X_t + \frac{1}{2}(u+\lambda_t)'\Sigma\Sigma'(u+\lambda_t) - \frac{1}{2}\lambda_t'\Sigma\Sigma'\lambda_t\right] \\
= & \exp\left[u'\Phi X_t + \frac{1}{2}u'\Sigma\Sigma'u + u'\Sigma\Sigma'\lambda_t\right] = \exp\left[u'\Phi X_t + \frac{1}{2}u'\Sigma\Sigma'u + u'\Sigma\Sigma'(\lambda_0 + \lambda_1'X_t)\right] \\
= & \exp\left[u'(\Phi + \Sigma\Sigma'\lambda_1')X_t + \frac{1}{2}u'\Sigma\Sigma'u + u'\Sigma\Sigma'\lambda_0\right],
\end{aligned}$$

which gives the result.

I.3 Proof of Proposition 2

Let us compute the Laplace transform of $Y_t = [X_t', Z_t', vec(X_tX_t')']'$. We have:

$$\begin{aligned}
& \mathbb{E}_t(\exp(u'Y_{t+1})) \\
= & \mathbb{E}_t(\exp(u_X'X_{t+1} + u_Z'Z_{t+1} + u_{XX}'vec(X_{t+1}X_{t+1}'))) \\
= & \mathbb{E}_t(\exp(u_X'(\Phi X_t + \Sigma\varepsilon_{t+1}) + u_Z'(\Phi_Z Z_t + \Sigma_Z(X_t)\varepsilon_{t+1}) + u_{XX}'vec((\Phi X_t + \Sigma\varepsilon_{t+1})(\Phi X_t + \Sigma\varepsilon_{t+1})'))) \\
= & \exp(u_X'\Phi X_t + u_Z'\Phi_Z Z_t + u_{XX}'vec(\Phi X_t X_t'\Phi')) \times \\
& \mathbb{E}_t\left(\exp\left[(u_X'\Sigma + u_Z'\Sigma_Z(X_t) + 2u_{XX}'(\Sigma\otimes[\Phi X_t]))\varepsilon_{t+1} + u_{XX}'vec(\Sigma\varepsilon_{t+1}\varepsilon_{t+1}'\Sigma')\right]\right), \qquad \text{(I.8)}
\end{aligned}$$

where we have used, in particular, that $vec(\Phi X_t\varepsilon_{t+1}'\Sigma') = (\Sigma\otimes[\Phi X_t])\varepsilon_{t+1}$ (exploiting the properties of the vec operator, see, e.g., Proposition A.1 of Monfort et al., 2015) and also $u_{XX}'vec(\Phi X_t\varepsilon_{t+1}'\Sigma') = u_{XX}'vec(\Sigma\varepsilon_{t+1}X_t'\Phi')$ (using the following lemma, since $vec^{-1}(u_{XX})$ is assumed to be a symmetric matrix).

Lemma 1. *If matrix V is symmetric, then, for any matrix A of the same dimension, we have $vec(V)'vec(A) = vec(V)'vec(A')$.*

Proof. For any matrix A, we have $vec(A') = \Lambda_n vec(A)$, where n is the dimension of V, and where Λ_n is the commutation matrix of dimension $n^2\times n^2$ (see Lemma A.1 in Monfort et al., 2015). In particular, since matrix $V$is symmetric, it comes that $vec(V) = \Lambda_n vec(V)$, or $vec(V)' = vec(V)'\Lambda_n'$. Using that Λ_n is orthogonal (and therefore that $\Lambda_n'\Lambda_n = I_n$) leads to the result. □

Moreover, we have:

$$\begin{aligned} u_{XX}' vec(\Sigma\varepsilon_{t+1}\varepsilon_{t+1}'\Sigma') &= u_{XX}'(\Sigma\otimes\Sigma)vec(\varepsilon_{t+1}\varepsilon_{t+1}') = u_{XX}'(\Sigma\otimes\Sigma)(\varepsilon_{t+1}\otimes\varepsilon_{t+1}) \\ &= (\varepsilon_{t+1}'\otimes\varepsilon_{t+1}')\{u_{XX}'(\Sigma\otimes\Sigma)\}' \\ &= \varepsilon_{t+1}' vec^{-1}\{u_{XX}'(\Sigma\otimes\Sigma)\}\varepsilon_{t+1}. \end{aligned} \tag{I.9}$$

Hence, the conditional expectation appearing at the end of (I.8) is of the form

$$\mathbb{E}_t(\exp[v(u,X_t)'\varepsilon_{t+1}+\varepsilon_{t+1}'V(u)\varepsilon_{t+1}]), \tag{I.10}$$

with

$$\left\{\begin{array}{lcl} v(u,X_t)' &=& u_X'\Sigma + u_Z'\Sigma_Z(X_t) + 2u_{XX}'(\Sigma\otimes[\Phi X_t]) \\ V(u) &=& vec^{-1}\{u_{XX}'(\Sigma\otimes\Sigma)\} \end{array}\right. \tag{I.11}$$

Let us show that $v(u,X_t)$ is affine in X_t. We have:

$$\begin{aligned} v(u,X_t) &= vec(v(u,X_t)') = vec(u_X'\Sigma + u_Z'\Sigma_Z(X_t) + 2u_{XX}'(\Sigma\otimes[\Phi X_t])) \\ &= \Sigma' u_X + (I_{n_\varepsilon}\otimes u_Z')(\Gamma_0+\Gamma_1 X_t) + 2(\Sigma\otimes[\Phi X_t])' u_{XX} \\ &= \Sigma' u_X + (I_{n_\varepsilon}\otimes u_Z')(\Gamma_0+\Gamma_1 X_t) + 2([\Phi X_t]'\otimes\Sigma') u_{XX} \\ &= \Sigma' u_X + (I_{n_\varepsilon}\otimes u_Z')(\Gamma_0+\Gamma_1 X_t) + 2vec(\Sigma' vec^{-1}(u_{XX})\Phi X_t) \\ &= \Sigma' u_X + (I_{n_\varepsilon}\otimes u_Z')\Gamma_0 + \left[(I_{n_\varepsilon}\otimes u_Z')\Gamma_1 + 2\Sigma' vec^{-1}(u_{XX})\Phi\right] X_t \\ &=: v_0(u) + v_1(u) X_t. \end{aligned} \tag{I.12}$$

Lemma 2. *If $\varepsilon \sim \mathcal{N}(0, I_{n_\varepsilon})$, and if v and V are, respectively, a n_ε-dimensional vector, and a $n_\varepsilon \times n_\varepsilon$ dimensional matrix, then*

$$\mathbb{E}(\exp[v'\varepsilon+\varepsilon' V\varepsilon]) = \frac{1}{|I_{n_\varepsilon}-2V|^{1/2}}\exp\left(\frac{1}{2}v'(I_{n_\varepsilon}-2V)^{-1}v\right).$$

Proof. See, e.g., Lemma A.2 in Dubecq et al. (2016). □

Using the previous lemma in (I.10), it comes that:

$$\begin{aligned} & \mathbb{E}_t(\exp[v(u,X_t)'\varepsilon_{t+1}+\varepsilon_{t+1}'V(u)\varepsilon_{t+1}]) \\ = & \frac{1}{|I_{n_\varepsilon}-2V|^{1/2}}\exp\left(\frac{1}{2}(v_0+v_1X_t)'(I_{n_\varepsilon}-2V)^{-1}(v_0+v_1X_t)\right) \\ = & \frac{1}{|I_{n_\varepsilon}-2V|^{1/2}}\exp\left(\frac{1}{2}(v_0'(I_{n_\varepsilon}-2V)^{-1}v_0 + X_t'v_1'(I_{n_\varepsilon}-2V)^{-1}v_1X_t + 2v_0'(I_{n_\varepsilon}-2V)^{-1}v_1X_t)\right), \end{aligned}$$

where, for notational simplicity, we have dropped the dependency in u of v_0, v_1, and V.

Substituting in the previous expression in (I.8), we obtain:

$$\begin{aligned} & \mathbb{E}_t(\exp(u'Y_{t+1})) \\ = & \exp(u_X'\Phi X_t + u_Z'\Phi_Z Z_t + X_t' vec^{-1}\{u_{XX}'(\Phi\otimes\Phi)\}X_t \times \end{aligned}$$

$$\exp\left[-\frac{1}{2}\log|I_{n_\varepsilon}-2V|+\frac{1}{2}v_0'(I_{n_\varepsilon}-2V)^{-1}v_0+\frac{1}{2}X_t'v_1'(I_{n_\varepsilon}-2V)^{-1}v_1X_t+v_0'(I_{n_\varepsilon}-2V)^{-1}v_1X_t\right],$$

which proves Prop. 2.

I.4 Proof of Proposition 4

According to Proposition 1, the s.d.f. is a function of X_{t+1} and X_t. In this context, Lemma 1 of Monfort and Renne (2013) implies that the risk-neutral p.d.f. of Z_{t+1} and w_{t+1}, given $(X_{t+1}, \underline{Y_t})$, are the same as the historical ones.

We have:

$$\begin{aligned} Z_t &= \Phi_Z Z_{t-1} + \Sigma_Z(X_{t-1})\varepsilon_t \\ &= \Phi_Z Z_{t-1} + \Sigma_Z(X_{t-1})\Sigma^{-1}(X_t - \Phi X_{t-1}) \\ &= \Sigma_Z(X_{t-1})\Sigma^{-1}\mu^{\mathbb{Q}} + \Phi_Z Z_{t-1} + \Sigma_Z(X_{t-1})\Sigma^{-1}(\Phi^{\mathbb{Q}}-\Phi)X_{t-1} + \Sigma_Z(X_{t-1})\varepsilon_t^{\mathbb{Q}}. \end{aligned} \tag{I.13}$$

Note that the previous formula is also valid if the dimension of X_t is lower than that of ε_t. In this case, however, one has to replace Σ^{-1} with the Moore-Penrose inverse of Σ.

We have:

$$\begin{aligned} \Sigma_Z(X_{t-1})\Sigma^{-1}\mu^{\mathbb{Q}} &= vec(\Sigma_Z(X_{t-1})\Sigma^{-1}\mu^{\mathbb{Q}}) \\ &= ([\Sigma^{-1}\mu^{\mathbb{Q}}]'\otimes I_{n_Z})vec(\Sigma_Z(X_{t-1})) \\ &= ([\Sigma^{-1}\mu^{\mathbb{Q}}]'\otimes I_{n_Z})(\Gamma_0+\Gamma_1 X_{t-1}). \end{aligned} \tag{I.14}$$

Moreover, we have:

$$\begin{aligned} \Sigma_Z(X_{t-1})\Sigma^{-1}(\Phi^{\mathbb{Q}}-\Phi)X_{t-1} &= vec(\Sigma_Z(X_{t-1})\Sigma^{-1}(\Phi^{\mathbb{Q}}-\Phi)X_{t-1}) \\ &= ([X_{t-1}'(\Sigma^{-1}(\Phi^{\mathbb{Q}}-\Phi))']\otimes I_{n_Z})(\Gamma_0+\Gamma_1 X_{t-1}) \end{aligned} \tag{I.15}$$

Let us denote by J_i the $n_\varepsilon \times (n_\varepsilon n_Z)$ matrix that selects the following n_ε entries of a vector Γ of dimension $(n_\varepsilon n_Z)\times 1$: $\{i, n_Z+i, \dots, (n_\varepsilon-1)n_Z+i\}$. That is $J_i = I_{n_\varepsilon}\otimes e_{i,n_Z}'$, where e_{i,n_Z} is the i^{th} column of the identity matrix of dimension $n_Z\times n_Z$.

For any matrix M of dimension $n_X \times n_\varepsilon$, we have:

$$((X_{t-1}'M)\otimes I_{n_Z})\Gamma_0 = \begin{bmatrix} (X_{t-1}'M)J_1\Gamma_0 \\ \vdots \\ (X_{t-1}'M)J_{n_Z}\Gamma_0 \end{bmatrix} = \begin{bmatrix} (\Gamma_0'J_1'M')X_{t-1} \\ \vdots \\ (\Gamma_0'J_{n_Z}'M')X_{t-1} \end{bmatrix} = \begin{bmatrix} \Gamma_0'J_1'M' \\ \vdots \\ \Gamma_0'J_{n_Z}'M' \end{bmatrix} X_{t-1} \tag{I.16}$$

Moreover:

$$((X_{t-1}'M)\otimes I_{n_Z})\Gamma_1 X_{t-1} = \begin{bmatrix} X_{t-1}'\Gamma_1'J_1'M'X_{t-1} \\ \vdots \\ X_{t-1}'\Gamma_1'J_{n_Z}'M'X_{t-1} \end{bmatrix} = \begin{bmatrix} vec(\Gamma_1'J_1'M')'vec(X_{t-1}X_{t-1}') \\ \vdots \\ vec(\Gamma_1'J_{n_Z}'M')'vec(X_{t-1}X_{t-1}') \end{bmatrix}$$

$$= \begin{bmatrix} vec(\Gamma_1' J_1' M')' \\ \vdots \\ vec(\Gamma_1' J_{n_Z}' M')' \end{bmatrix} vec(X_{t-1} X_{t-1}'). \tag{I.17}$$

Using (I.16) and (I.17) in (I.15), we obtain:

$$\begin{aligned} & \Sigma_Z(X_{t-1})\Sigma^{-1}(\Phi^{\mathbb{Q}} - \Phi)X_{t-1} \\ = & \underbrace{\begin{bmatrix} \Gamma_0' J_1' \Sigma^{-1}(\Phi^{\mathbb{Q}} - \Phi) \\ \vdots \\ \Gamma_0' J_{n_Z}' \Sigma^{-1}(\Phi^{\mathbb{Q}} - \Phi) \end{bmatrix}}_{=:\widetilde{\Gamma}_0} X_{t-1} + \underbrace{\begin{bmatrix} vec(\Gamma_1' J_1' \Sigma^{-1}(\Phi^{\mathbb{Q}} - \Phi))' \\ \vdots \\ vec(\Gamma_1' J_{n_Z}' \Sigma^{-1}(\Phi^{\mathbb{Q}} - \Phi))' \end{bmatrix}}_{=:\widetilde{\Gamma}_1} vec(X_{t-1} X_{t-1}') \end{aligned} \tag{I.18}$$

Using (I.14) and (I.18) in (I.13) leads to (35) and proves Prop. 4.

I.5 Proof of Proposition 5

We have:

$$\begin{aligned} & \mathbb{E}_t^{\mathbb{Q}}(\exp(u'Y_{t+1})) \\ = & \mathbb{E}_t^{\mathbb{Q}}(\exp(u_X' X_{t+1} + u_Z' Z_{t+1} + u_{XX}' vec[X_{t+1} X_{t+1}'])) \\ = & \mathbb{E}_t^{\mathbb{Q}}(\exp(u_X'(\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_t + \Sigma \varepsilon_{t+1}^{\mathbb{Q}}) + \\ & u_Z'\{\mu_Z^{\mathbb{Q}} + \Phi_Z Z_t + \Phi_{ZX}^{\mathbb{Q}} X_t + \Phi_{ZXX}^{\mathbb{Q}} vec(X_t X_t') + \Sigma_Z(X_t)\varepsilon_{t+1}^{\mathbb{Q}}\} + u_{XX}' vec[X_{t+1} X_{t+1}'])) \\ = & \exp(u_X'\mu^{\mathbb{Q}} + u_X'\Phi^{\mathbb{Q}} X_t + u_Z'\mu_Z^{\mathbb{Q}} + u_Z'\Phi_Z Z_t + u_Z'\Phi_{ZX}^{\mathbb{Q}} X_t + u_Z'\Phi_{ZXX}^{\mathbb{Q}} vec(X_t X_t')) \times \\ & \mathbb{E}_t^{\mathbb{Q}}(\exp(u_X'\Sigma\varepsilon_{t+1}^{\mathbb{Q}} + u_Z'\Sigma_Z(X_t)\varepsilon_{t+1}^{\mathbb{Q}} + u_{XX}' vec[X_{t+1} X_{t+1}'])). \end{aligned} \tag{I.19}$$

Let us focus on the last term:

$$\begin{aligned} vec[X_{t+1} X_{t+1}'] &= vec[(\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_t + \Sigma\varepsilon_{t+1}^{\mathbb{Q}})(\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_t + \Sigma\varepsilon_{t+1}^{\mathbb{Q}})'] \\ &= vec\Big(\mu^{\mathbb{Q}}\mu^{\mathbb{Q}\prime} + \mu^{\mathbb{Q}} X_t' \Phi^{\mathbb{Q}\prime} + \Phi^{\mathbb{Q}} X_t \mu^{\mathbb{Q}\prime} + \mu^{\mathbb{Q}} \varepsilon_{t+1}^{\mathbb{Q}\,\prime}\Sigma' + \Sigma\varepsilon_{t+1}^{\mathbb{Q}}\mu^{\mathbb{Q}\prime} \\ &\quad + \Phi^{\mathbb{Q}} X_t X_t' \Phi^{\mathbb{Q}\prime} + \Phi^{\mathbb{Q}} X_t \varepsilon_{t+1}^{\mathbb{Q}\,\prime}\Sigma' + \Sigma\varepsilon_{t+1}^{\mathbb{Q}} X_t' \Phi^{\mathbb{Q}\prime} + \Sigma\varepsilon_{t+1}^{\mathbb{Q}}\varepsilon_{t+1}^{\mathbb{Q}\,\prime}\Sigma'\Big) \\ &= vec(\mu^{\mathbb{Q}}\mu^{\mathbb{Q}\prime}) + (I_{n_X^2} + \Lambda_{n_X})(\Phi^{\mathbb{Q}} \otimes \mu^{\mathbb{Q}})X_t + (\Phi^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}}) vec(X_t X_t') \\ &\quad + (I_{n_X^2} + \Lambda_{n_X})(\Sigma \otimes [\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_t])\varepsilon_{t+1}^{\mathbb{Q}} + (\Sigma \otimes \Sigma) vec(\varepsilon_{t+1}^{\mathbb{Q}}\varepsilon_{t+1}^{\mathbb{Q}\,\prime}), \end{aligned}$$

where Λ_{n_X} is the commutation matrix of dimension $n_X^2 \times n_X^2$ (see the proof of Lemma 1). Since $vec^{-1}(u_{XX})$ is a $n_X \times n_X$ symmetric matrix (by assumption), it comes that $u_{XX} = \Lambda_{n_X} u_{XX}$. As a result:

$$\begin{aligned} u_{XX}' vec[X_{t+1} X_{t+1}'] &= u_{XX}' vec(\mu^{\mathbb{Q}}\mu^{\mathbb{Q}\prime}) + 2u_{XX}'(\Phi^{\mathbb{Q}} \otimes \mu^{\mathbb{Q}})X_t + u_{XX}'(\Phi^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}}) vec(X_t X_t') \\ &\quad + 2u_{XX}'(\Sigma \otimes [\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_t])\varepsilon_{t+1}^{\mathbb{Q}} + u_{XX}'(\Sigma \otimes \Sigma) vec(\varepsilon_{t+1}^{\mathbb{Q}}\varepsilon_{t+1}^{\mathbb{Q}\,\prime}). \end{aligned}$$

Using the previous expression in (I.19), we obtain:

$$
\begin{aligned}
&\mathbb{E}_t^{\mathbb{Q}}(\exp(u'Y_{t+1})) = \\
&\exp\{u_X'\mu^{\mathbb{Q}} + u_{XX}'vec(\mu^{\mathbb{Q}}\mu^{\mathbb{Q}\prime}) + u_Z'\mu_Z^{\mathbb{Q}} \\
&+(u_X'\Phi^{\mathbb{Q}} + 2u_{XX}'(\Phi^{\mathbb{Q}}\otimes\mu^{\mathbb{Q}}) + u_Z'\Phi_{ZX}^{\mathbb{Q}})X_t + u_Z'\Phi_Z Z_t \\
&+(u_Z'\Phi_{ZXX}^{\mathbb{Q}} + u_{XX}'(\Phi^{\mathbb{Q}}\otimes\Phi^{\mathbb{Q}}))vec(X_tX_t')\}\times \\
&\mathbb{E}_t^{\mathbb{Q}}\left(\exp\left(\left[u_X'\Sigma + 2u_{XX}'(\Sigma\otimes\mu^{\mathbb{Q}}) + 2u_{XX}'(\Sigma\otimes[\Phi^{\mathbb{Q}}X_t]) + u_Z'\Sigma_Z(X_t)\right]\varepsilon_{t+1}^{\mathbb{Q}} + u_{XX}'(\Sigma\otimes\Sigma)vec(\varepsilon_{t+1}^{\mathbb{Q}}\varepsilon_{t+1}^{\mathbb{Q}\ \prime})\right)\right).
\end{aligned}
$$

The last conditional expectation is similar to that appearing in (I.8); that is, it is of the form:

$$
\mathbb{E}_t^{\mathbb{Q}}\left(v^*(u,X_t)'\varepsilon_{t+1}^{\mathbb{Q}} + \varepsilon_{t+1}^{\mathbb{Q}\ \prime}V(u)\varepsilon_{t+1}^{\mathbb{Q}}\right), \tag{I.20}
$$

with

$$
v^*(u,X_t)' = u_X'\Sigma + 2u_{XX}'(\Sigma\otimes\mu^{\mathbb{Q}}) + 2u_{XX}'(\Sigma\otimes[\Phi^{\mathbb{Q}}X_t]) + u_Z'\Sigma_Z(X_t),
$$

and $V(u)$ is as in (I.11).

We can proceed as in (I.12) to show that $v^*(u,X_t)$ is affine in X_t. This leads to

$$
v^*(u,X_t) = v_0^*(u) + v_1^*(u)X_t,
$$

with

$$
\left\{\begin{array}{lcl}
v_0^*(u) & = & \Sigma' u_X + 2(\Sigma'\otimes\mu^{\mathbb{Q}\prime})u_{XX} + (I_{n_\varepsilon}\otimes u_Z')\Gamma_0 \\
v_1^*(u) & = & (I_{n_\varepsilon}\otimes u_Z')\Gamma_1 + 2\Sigma' vec^{-1}(u_{XX})\Phi^{\mathbb{Q}}.
\end{array}\right. \tag{I.21}
$$

We can apply Lemma 2 to evaluate (I.20). This leads to the result presented in Prop. 5.

I.6 Proof of Prop. 7

The price of a h-period nominal bond is given by

$$
\begin{aligned}
P_{t,h}^{\$} &= \mathbb{E}_t(\mathscr{M}_{t,t+h}\exp(-\pi_{t+1}-\cdots-\pi_{t+h})) \\
&= \mathbb{E}_t\left(\frac{\mathscr{M}_{t,t+h}}{\mathbb{E}_t(\mathscr{M}_{t,t+h})}\mathbb{E}_t(\mathscr{M}_{t,t+h})\exp(-\pi_{t+1}-\cdots-\pi_{t+h})\right) \\
&= \mathbb{E}_t^{\mathbb{Q}}\left(\mathbb{E}_t(\mathscr{M}_{t,t+h})\exp(-\pi_{t+1}-\cdots-\pi_{t+h})\right) \\
&= \mathbb{E}_t^{\mathbb{Q}}(\exp(-r_t-\cdots-r_{t+h-1}-\pi_{t+1}-\cdots-\pi_{t+h})).
\end{aligned}
$$

We have:

$$
\begin{aligned}
P_{t,h+1}^{\$} &= \mathbb{E}_t^{\mathbb{Q}}\left(\exp(-r_t-\pi_{t+1})P_{t+1,h}^{\$}\right) \\
&= \mathbb{E}_t^{\mathbb{Q}}\left(\exp\left[-(\eta_0^* + \eta_1^{*\prime}X_t) - \mu_{\pi,0} - \mu_{\pi,Z}'Z_{t+1} - \mu_{\pi,X}'X_{t+1} - \mu_{\pi,XX}'vec(X_{t+1}X_{t+1}')\right.\right. \\
&\quad \left.\left. + a_h^{\$} + b_h^{\$\prime}X_{t+1} + c_h^{\$\prime}Z_{t+1} + d_h^{\$\prime}vec(X_{t+1}X_{t+1}')\right]\right) \\
&= \exp(-\eta_0^* - \eta_1^{*\prime}X_t - \mu_{\pi,0} + a_h^{\$})\times
\end{aligned}
$$

$$\begin{aligned}
& \mathbb{E}_t^{\mathbb{Q}}\left(\exp\left[(b_h^{\$}-\mu_{\pi,X})'X_{t+1}+\left(c_h^{\$}-\mu_{\pi,Z}\right)'Z_{t+1}+(d_h^{\$}-\mu_{\pi,XX})'vec(X_{t+1}X_{t+1}')\right]\right)\\
= \; & \exp(-\eta_0^*-\eta_1^{*\prime}X_t-\mu_{\pi,0}+a_h^{\$})\times \\
& \exp\left[\psi_{Y,0}^{\mathbb{Q}}(u_h)+\psi_{Y,X}^{\mathbb{Q}}(u_h)'X_t+\psi_{Y,Z}^{\mathbb{Q}}(u_h)'Z_t+\psi_{Y,XX}^{\mathbb{Q}}(u_h)'vec(X_tX_t')\right],
\end{aligned}$$

where u_h is as defined in (41). This gives the result.

II First- and second-order moments of Y_t

In this section, we derive the first-order and second-order moments of Y_t. We consider conditional moments, say $\mathbb{E}_t(Y_{t+1})$, and unconditional moments, as $\mathbb{E}(Y_t)$.

Proposition 9. *Under Assumptions A.2 and A.3, the conditional expectation and variance of process Y_t (with $Y_t=[X_t',Z_t',vech(X_tX_t')']'$) are given by:*

$$\mathbb{E}_t(Y_{t+1}) = \left[\frac{\partial}{\partial u}\psi_{Y,0}(u)\right]_{u=0}+\left[\frac{\partial}{\partial u}\psi_{Y,1}(u)'\right]_{u=0}Y_t \tag{II.1}$$

$$vec\left(\mathbb{V}ar_t(Y_{t+1})\right) = \Theta_0+\Theta_1 Y_t, \tag{II.2}$$

where

$$\Theta_0=\left[\frac{\partial^2}{\partial u\partial u'}\psi_{Y,0}(u)\right]_{u=0},$$

and where Θ_1 is such that its $((i-1)n_Y+j)^{th}$ row is:

$$\left[\frac{\partial^2}{\partial u_i\partial u_j}\psi_{Y,1}(u)'\right]_{u=0}.$$

Proof. Since we have:

$$\frac{\partial}{\partial u}\mathbb{E}_t(\exp(u'Y_{t+1}))=\mathbb{E}_t(Y_{t+1}\exp(u'Y_{t+1})),$$

it comes that

$$\frac{\partial}{\partial u}\mathbb{E}_t(\exp(u'Y_{t+1}))\Big|_{u=0}=\mathbb{E}_t(Y_{t+1}).$$

Now, according to Prop. 2, we have $\mathbb{E}_t(\exp(u'Y_{t+1}))=\exp\left(\psi_{Y,0}(u)+\psi_{Y,1}(u)'Y_t\right)$. Hence, we also have:

$$\left[\frac{\partial}{\partial u}\mathbb{E}_t(\exp(u'Y_{t+1}))\right]_{u=0}=\left[\frac{\partial}{\partial u}\psi_{Y,0}(u)\right]_{u=0}+\left[\frac{\partial}{\partial u}\psi_{Y,1}(u)'\right]_{u=0}Y_t,$$

using, in particular, that $\psi_{Y,0}(0)=0$ and $\psi_{Y,1}(0)=0$ since $\mathbb{E}_t(\exp(0'Y_{t+1}))=1$. The variance result is obtained in a similar way. □

Corollary 1. *Under Assumptions A.2 and A.3, the dynamics of Y_t admits the following vector auto-regressive representation:*

$$Y_{t+1} = \mu_Y + \Phi_Y Y_t + \Sigma_Y^{\frac{1}{2}}(Y_t)\varepsilon_{Y,t+1}, \tag{II.3}$$

where $\varepsilon_{Y,t+1}$ is a martingale difference sequence satisfying $\mathbb{V}ar_t(\varepsilon_{Y,t+1}) = I_{n_Y}$ (identity matrix of dimension $n_Y \times n_Y$), and where

$$\begin{aligned} \mu_Y = \left[\frac{\partial}{\partial u}\psi_{Y,0}(u)\right]_{u=0}, \quad \Phi_Y = \left[\frac{\partial}{\partial u}\psi_{Y,1}(u)'\right]_{u=0}, \\ vec(\Sigma_Y(Y_t)) := vec\left[\Sigma_Y^{\frac{1}{2}}(Y_t)\Sigma_Y^{\frac{1}{2}}(Y_t)'\right] = \Theta_0 + \Theta_1 Y_t, \end{aligned}$$

Θ_0 and Θ_1 being given in Prop. 9.

Corollary 2. *Under Assumptions A.2 and A.3, we have:*

$$\begin{aligned} \mathbb{E}_t(Y_{t+h}) &= (I_{n_Y} - \Phi_Y)^{-1}(I_{n_Y} - \Phi_Y^h)\mu_Y + \Phi_Y^h Y_t \\ vec\left[\mathbb{V}ar_t(Y_{t+h})\right] &= vec\left[\Sigma_Y(Y_t) + \Phi_Y\Sigma_Y(Y_t)\Phi_Y' + \dots + \Phi_Y^{h-1}\Sigma_Y(Y_t)\Phi_Y^{h-1\prime}\right] \\ &= \left(I_{n_\varepsilon^2} + \Phi_Y \otimes \Phi_Y + \dots + \Phi_Y^{h-1} \otimes \Phi_Y^{h-1}\right)(\Theta_0 + \Theta_1 Y_t) \\ &= \left(I_{n_\varepsilon^2} + \Phi_Y \otimes \Phi_Y + \dots + (\Phi_Y \otimes \Phi_Y)^{h-1}\right)(\Theta_0 + \Theta_1 Y_t) \\ &= \left(I_{n_\varepsilon^2} - \Phi_Y \otimes \Phi_Y\right)^{-1}\left(I_{n_\varepsilon} - (\Phi_Y \otimes \Phi_Y)^h\right)(\Theta_0 + \Theta_1 Y_t). \end{aligned}$$

Corollary 3. *Under Assumptions A.2 and A.3, we have:*

$$\begin{aligned} \mathbb{E}(Y_t) &= (I_{n_Y} - \Phi_Y)^{-1}\mu_Y \\ vec[\mathbb{V}ar(Y_t)] &= \left(I_{n_Y^2} - \Phi_Y \otimes \Phi_Y\right)^{-1} vec\left[\Sigma_Y\left\{(Id - \Phi_Y)^{-1}\mu_Y\right\}\right]. \end{aligned}$$

Corollary 4. *Under Assumptions A.2 and A.3, we have:*

$$\mathbb{C}ov(Y_t, Y_{t+h}) = \mathbb{V}ar(Y_t)(\Phi_Y^h)'.$$

Proof. We have:

$$\begin{aligned} \mathbb{C}ov(Y_t, Y_{t+h}) &= \mathbb{E}(Y_t Y_{t+h}') - \mathbb{E}(Y_t)\mathbb{E}(Y_t)' = \mathbb{E}(Y_t \mathbb{E}_t(Y_{t+h})') - \mathbb{E}(Y_t)\mathbb{E}(Y_t)' \\ &= \mathbb{E}(Y_t[(I_{n_Y} - \Phi_Y)^{-1}(I_{n_Y} - \Phi_Y^h)\mu_Y + \Phi_Y^h Y_t]') - \mathbb{E}(Y_t)\mathbb{E}(Y_t)' \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}(Y_t \mu_Y' (I_{n_Y} - \Phi_Y^h)'(I_{n_Y} - \Phi_Y')^{-1} + Y_t Y_t' (\Phi_Y^h)') - \mathbb{E}(Y_t)\mathbb{E}(Y_t)' \\
&= (I_{n_Y} - \Phi_Y)^{-1} \mu_Y \mu_Y' (I_{n_Y} - \Phi_Y^h)'(I_{n_Y} - \Phi_Y')^{-1} + \mathbb{E}(Y_t Y_t')(\Phi_Y^h)' - \mathbb{E}(Y_t)\mathbb{E}(Y_t)' \\
&= (I_{n_Y} - \Phi_Y)^{-1} \mu_Y \mu_Y' (I_{n_Y} - \Phi_Y')^{-1}(I_{n_Y} - \Phi_Y^h)' + \mathbb{E}(Y_t Y_t')(\Phi_Y^h)' - \mathbb{E}(Y_t)\mathbb{E}(Y_t)' \\
&= \mathbb{E}(Y_t)\mathbb{E}(Y_t)'(I_{n_Y} - \Phi_Y^h)' + \mathbb{E}(Y_t Y_t')(\Phi_Y^h)' - \mathbb{E}(Y_t)\mathbb{E}(Y_t)',
\end{aligned}
$$

which gives the result. □

III Real term premiums in the Consumption Capital Asset Pricing Model

III.1 Term premiums and conditional covariances of the SDF

The term premium of maturity n is defined as the difference between the yield-to-maturity of a bond of maturity n and the one that would prevail under the expectation hypothesis. That is:

$$TP_{t,n} = -\frac{1}{n}\log \mathbb{E}_t \mathscr{M}_{t,t+n} + \frac{1}{n}\log \mathbb{E}_t \exp(-r_t - r_{t+1} - \dots - r_{t+n-1}).$$

The following proposition shows how term premiums can be expressed as a conditional covariance involving future stochastic discount factors (SDFs) and their expectations.

Proposition 10. *If the log SDF $m_{t+1} = \log(\mathscr{M}_{t,t+1})$ is Gaussian and homoskedastic, we have:*

$$\begin{aligned} TP_{t,n} &= -\frac{1}{n}\mathbb{C}ov_t(m_{t+1} - \mathbb{E}_t(m_{t+1}) + \dots + m_{t+n-1} - \mathbb{E}_{t+n-2}(m_{t+n-1}), \\ &\quad \mathbb{E}_{t+1}(m_{t+2}) + \dots + \mathbb{E}_{t+n-1}(m_{t+n})). \end{aligned}$$

Proof. We have

$$TP_{t,n} := -\frac{1}{n}\log \mathbb{E}_t(\exp(m_{t+1} + \dots + m_{t+n})) + \frac{1}{n}\log \mathbb{E}_t(\exp(-r_t - \dots - r_{t+n-1})).$$

Since m_{t+1} is Gaussian, we have

$$\exp(-r_t) = \mathbb{E}_t \exp(m_{t+1}) = \mathbb{E}_t(m_{t+1}) + \frac{1}{2}\mathbb{V}ar_t(m_{t+1}).$$

As a consequence:

$$\begin{aligned} TP_{t,n} &= -\frac{1}{n}\log \mathbb{E}_t(\exp(m_{t+1} + \dots + m_{t+n})) + \\ &\quad \frac{1}{n}\log \mathbb{E}_t\left(\exp\left(\mathbb{E}_t(m_{t+1}) + \frac{1}{2}\mathbb{V}ar_t(m_{t+1}) + \dots + \mathbb{E}_{t+n-1}(m_{t+n}) + \frac{1}{2}\mathbb{V}ar_{t+n-1}(m_{t+n})\right)\right). \end{aligned}$$

Under homoskedasticity, we have $\mathbb{V}ar_t(m_{t+1}) = \sigma_m^2$, say, for any t. This gives:

$$\begin{aligned} TP_{t,n} &= \frac{1}{2}\sigma_m^2 - \frac{1}{n}\log \mathbb{E}_t(\exp(m_{t+1} + \dots + m_{t+n})) + \\ &\quad \frac{1}{n}\log \mathbb{E}_t\left(\exp\left(\mathbb{E}_t(m_{t+1}) + \dots + \mathbb{E}_{t+n-1}(m_{t+n})\right)\right). \end{aligned}$$

Using that $m_{t+1} + \dots + m_{t+n}$ is Gaussian , we obtain:

$$TP_{t,n} = \frac{1}{2}\sigma_m^2 - \frac{1}{2n}\mathbb{V}ar_t\left[m_{t+1} + \dots + m_{t+n}\right] + \frac{1}{2n}\mathbb{V}ar_t\left[\mathbb{E}_t(m_{t+1}) + \dots + \mathbb{E}_{t+n-1}(m_{t+n})\right].$$

Let us focus on $\mathbb{V}ar_t\,[m_{t+1}+\cdots+m_{t+n}]$. Since $m_{t+1}=[m_{t+1}-\mathbb{E}_t(m_{t+1})]+\mathbb{E}_t(m_{t+1})$, we get:

$$\begin{aligned} & \mathbb{V}ar_t\,[m_{t+1}+\cdots+m_{t+n}] \\ = \; & \mathbb{V}ar_t\,[\{m_{t+1}-\mathbb{E}_t(m_{t+1})\}+\cdots+\{m_{t+n}-\mathbb{E}_{t+n-1}(m_{t+n})\}] \\ & +\mathbb{V}ar_t\,[\mathbb{E}_t(m_{t+1})+\cdots+\mathbb{E}_{t+n-1}(m_{t+n})] \\ & +2\mathbb{C}ov_t(m_{t+1}-\mathbb{E}_t(m_{t+1})+\cdots+m_{t+n}-\mathbb{E}_{t+n-1}(m_{t+n}),\mathbb{E}_t(m_{t+1})+\cdots+\mathbb{E}_{t+n-1}(m_{t+n})). \end{aligned}$$

Using $\mathbb{V}ar_t\,[\{m_{t+1}-\mathbb{E}_t(m_{t+1})\}+\cdots+\{m_{t+n}-\mathbb{E}_{t+n-1}(m_{t+n})\}]=n\sigma_m^2$ leads to the result. □

Proposition 10 implies in particular, for $n=2$:

$$TP_{t,2} \;=\; -\frac{1}{n}\mathbb{C}ov_t(m_{t+1},\mathbb{E}_{t+1}(m_{t+2})). \tag{III.1}$$

Hence, to have a positive two-period real term premium, we must have $\mathbb{C}ov_t(m_{t+1},\mathbb{E}_{t+1}(m_{t+2}))<0$.

III.2 A simple trend-cycle decomposition of consumption

Consider a simplified version of the model developed in the paper, with the aim of exploring analytically the slope of the term structure of real term premiums. Assume that date-t consumption, denoted by C_t, is given by:

$$C_t=C_t^*\exp(z_t),$$

where C_t^* can be interpreted as the consumption trend and z_t is its cyclical component (or output gap). Using small letters for logarithms, we get:

$$c_t=c_t^*+z_t.$$

Denoting the trend growth rate by g_t, i.e.,

$$g_t=c_t^*-c_{t-1}^*=\Delta c_t^*,$$

we obtain:

$$\Delta c_t=g_t+z_t-z_{t-1}.$$

Assume that both g_t and z_t follow auto-regressive processes of order one:

$$\begin{aligned} g_t &= (1-\rho_g)\mu_g+\rho_g g_{t-1}+\eta_t \\ z_t &= \rho_z z_{t-1}+\nu_t, \end{aligned}$$

where $\eta_t\sim i.i.d.\,\mathcal{N}(0,\sigma_g)$ and $\nu_t\sim i.i.d.\,\mathcal{N}(0,\sigma_z)$ (Note that what precedes implies, in particular, that $\mathbb{E}(\Delta c_t)=\mu_g$.)

For simplicity, we replace the Epstein-Zin preferences used in the paper with power-utility time-separable preferences. In that case, as is well-known, the stochastic discount factor between dates t and $t+1$ is given by:

$$\mathscr{M}_{t,t+1} = \delta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} = \exp[\log(\delta) - \gamma \Delta c_{t+1}],$$

and, therefore, $m_{t+1} = \log \mathscr{M}_{t,t+1} = \log(\delta) - \gamma \Delta c_{t+1}$. In this context, Proposition 10 implies (this is eq. III.1):

$$TP_{t,2} = -\frac{\gamma^2}{2} \mathbb{C}ov_t[\Delta c_{t+1}, \mathbb{E}_{t+1}(\Delta c_{t+1})].$$

Since

$$\mathbb{C}ov_t[\Delta c_{t+1}, \mathbb{E}_{t+1}(\Delta c_{t+1})] = \rho_g \sigma_g^2 + (\rho_z - 1)\sigma_z^2, \quad \text{(III.2)}$$

we obtain:

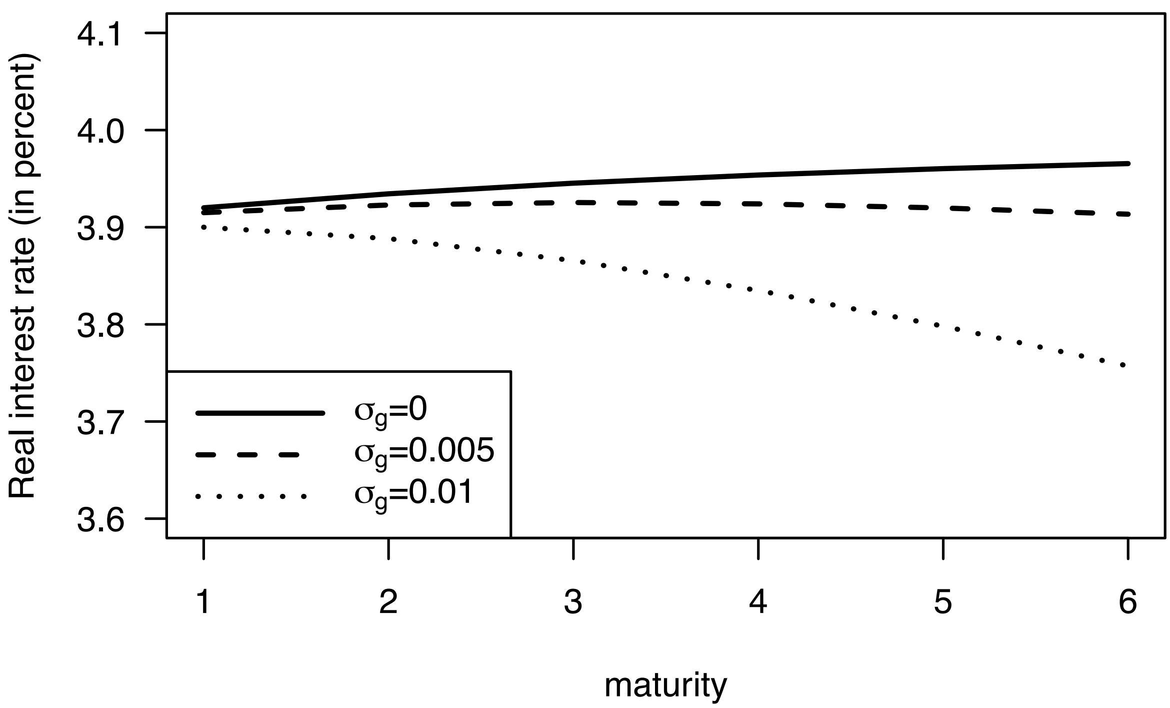
$$TP_{t,2} = \frac{\gamma^2}{2}[-\rho_g \sigma_g^2 + (1-\rho_z)\sigma_z^2],$$

which is positive if:

$$(1-\rho_z)\sigma_z^2 > \rho_g \sigma_g^2,$$

that is, if the contribution of the cyclical component dominates in $\mathbb{C}ov_t[\Delta c_{t+1}, \mathbb{E}_{t+1}(\Delta c_{t+1})]$.

Figure III.1: Average term structure of real interest rates



Notes: Term structures of real interest rates obtained for $x_t = [\mu_g, 0, 0]'$, with $\mu_g = 0.02$, $\gamma = 2$, $\delta = 1$, $\rho_g = 0.9$, $\rho_z = 0.8$, $\sigma_z = 0.02$, and different values of σ_g. The pricing formulas are those given in Proposition 11.

Figure III.1 extends this analysis for long horizons (using the pricing formulas of Proposition 10). It confirms that the real term premium real term premiums can be upward sloping if σ_g is small enough compared to σ_z (at least between horizons one and two). In particular, if $\sigma_g^2 = 0$, the term premium is positive. In that case, agents know that, if a bad state of the world materializes in the next date (i.e., $\nu_{t+1} < 0$), then the expected one-period-ahead growth, as of date $t+1$, will then be positive—since agents will then expect the output gap to close—which will translate into a higher r_{t+1}. Alternatively put, as of date t, agents know that $P_{t+1,1}$ will decrease in bad states of the world, and vice versa. This implies that a two-period bond does not hedge against bad states of the world, which generates a positive term premium.

In several papers that consider term structures of real rates in a structural framework, consumption growth is essentially based on autoregressive processes akin to g_t (this is notably the case when the model only consider a stochastic autoregressive productivity process). Hence, in these contexts, the term structure of real rates is necessarily downward sloping.

Proposition 11. *The model described in Appendix III.2 can be cast into a VAR form:*

$$x_t = \begin{bmatrix} g_t \\ z_t \\ z_{t-1} \end{bmatrix} = \underbrace{\begin{bmatrix} \mu_g(1-\rho_g) \\ 0 \\ 0 \end{bmatrix}}_{=\mu} + \underbrace{\begin{bmatrix} \rho_g & 0 & 0 \\ 0 & \rho_z & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{=\Phi} \begin{bmatrix} g_{t-1} \\ z_{t-1} \\ z_{t-2} \end{bmatrix} + \underbrace{\begin{bmatrix} \eta_t \\ \nu_t \\ 0 \end{bmatrix}}_{=\varepsilon_t}. \qquad \text{(III.3)}$$

Using the previous notations, the price of a (real) zero-coupon bond of maturity h is given by

$$P_{t,h} = \exp(a_h + b_h' x_t),$$

where

$$\begin{cases} a_{h+1} &= \log(\delta) + a_h + (b_h - \gamma\alpha)'\mu + \frac{1}{2}(b_h - \gamma\alpha)'\Omega(b_h - \gamma\alpha) \\ b_{h+1} &= \Phi'(b_h - \gamma\alpha), \end{cases}$$

with

$$\Omega = \begin{bmatrix} \sigma_g^2 & 0 & 0 \\ 0 & \sigma_z^2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and $a_0 = 0$, $b_0 = 0$.

Proof. With the notation introduced in (III.3), we have:

$$\Delta c_t = \underbrace{\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}}_{=\alpha'} x_t,$$

and, therefore,

$$\mathscr{M}_{t,t+1} = \exp(\log(\delta) - \gamma\alpha' x_{t+1}).$$

Hence:

$$\begin{aligned} P_{t,h+1} &= \mathbb{E}_t(\mathscr{M}_{t,t+1} P_{t+1,h}) = \mathbb{E}_t(\exp[\log(\delta) - \gamma\alpha' x_{t+1} + a_h + b_h' x_{t+1}]) \\ &= \mathbb{E}_t(\exp[\log(\delta) + a_h + (b_h - \gamma\alpha) x_{t+1}]) \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_t(\exp[\log(\delta)+a_h+(b_h-\gamma\alpha)'(\mu+\Phi x_t+\varepsilon_{t+1})]) \\
&= \exp[\log(\delta)+a_h+(b_h-\gamma\alpha)'(\mu+\Phi x_t)]\mathbb{E}_t(\exp[(b_h-\gamma\alpha)'\varepsilon_{t+1}]) \\
&= \exp\left[\log(\delta)+a_h+(b_h-\gamma\alpha)'(\mu+\Phi x_t)+\frac{1}{2}(b_h-\gamma\alpha)'\Omega(b_h-\gamma\alpha)\right],
\end{aligned}
$$

which leads to the result. □

III.3 About the implications of the trend/cycle specification on consumption moments

This appendix discusses some implications that the trend-cycle representation of consumption growth has on the conditional moments of consumption.

We have seen that, in the context of the model described in Appendix III, we have (this is eq. III.2):

$$\mathbb{C}ov_t[\Delta c_{t+1},\mathbb{E}_{t+1}(\Delta c_{t+1})] = \rho_g\sigma_g^2+(\rho_z-1)\sigma_z^2.$$

Besides, since

$$\mathbb{V}ar_t(\Delta c_{t+1}) = \sigma_g^2+\sigma_z^2, \quad \text{and} \quad \mathbb{V}ar_t(\mathbb{E}_{t+1}\Delta c_{t+2}) = \rho_g^2\sigma_g^2+(\rho_z-1)^2\sigma_z^2,$$

it comes that:

$$\mathbb{C}orr_t[\Delta c_{t+1},\mathbb{E}_{t+1}(\Delta c_{t+1})] = \frac{\rho_g\sigma_g^2+(\rho_z-1)\sigma_z^2}{\sqrt{\sigma_g^2+\sigma_z^2}\sqrt{\rho_g^2\sigma_g^2+(1-\rho_z)^2\sigma_z^2}}.$$

It can be noteed that this correlation takes extreme values when $\sigma_g = 0$, in which case it is equal to -1, and when $\sigma_z = 0$, in which case it is equal to 1. If direct data-based counterparts of this *conditional* correlation were available, one could determine whether one of these two extreme representations has to be discarded; but this is not the case. One can nevertheless exploit surveys to determine *unconditional* correlations between output growth and expected output growth. The model-implied unconditional correlation is given by:

$$\mathbb{C}orr(\Delta c_{t+1},\mathbb{E}_{t+1}(\Delta c_{t+1})) = \frac{\frac{\rho_g}{1-\rho_g^2}\sigma_g^2-\frac{1-\rho_z}{1+\rho_z}\sigma_z^2}{\sqrt{\frac{1}{1-\rho_g^2}\sigma_g^2+\frac{2}{1+\rho_z}\sigma_z^2}\sqrt{\frac{\rho_g^2}{1-\rho_g^2}\sigma_g^2+\frac{1-\rho_z}{1+\rho_z}\sigma_z^2}}.$$

It can be seen that, when $\sigma_z = 0$, this correlation is still extreme, as it is equal to one. But it is not extreme for $\sigma_g = 0$. In the latter case, we have:

$$\mathbb{C}orr(\Delta c_{t+1},\mathbb{E}_{t+1}(\Delta c_{t+1})) = -\sqrt{\frac{1-\rho_z}{2}}.$$

Let us consider empirical estimates of this correlation. For that, we use the real GDP series available on the FRED database (ticker GDPC1), as well as the mean GDP forecasts of professional forecasters extracted from the website of the Philadelphia Federal Reserve Bank. Using these data, we can compute the empirical correlation between the quarterly output growth ($\log(Y_t/Y_{t-1})$) and expected output growth ($\mathbb{E}_t(\log(Y_{t+1}/Y_t))$). We also consider an annual version, where we compute the correlation between $\log(Y_t/Y_{t-4})$ and $\mathbb{E}_t(\log(Y_{t+4}/Y_t))$. Table III.1 shows the resulting correlations for two periods: a long historical period (1968-2024), and a shorter one (1994-2024). The last row of the table reports the model-implied equivalent correlations (using the model presented in Section 2 and whose specification is detailed in Table 2).

Table III.1: Unconditional correlations between current and expected output growth

Period	**Quarterly**	**Annual**
1968Q4-2024Q1	27%	17%
1994Q1-2024Q1	−4%	7%
Model	−14%	−7%

Notes: This table shows the correlations between output growth ($\log(Y_t/Y_{t-k})$) and expected output growth ($\mathbb{E}_t(\log(Y_{t+k}/Y_t))$). For column "Quarterly", we use $k = 1$, and $k = 4$ for column "Annual". Real GDP comes from the FRED database (ticker GDPC1); GDP forecasts are extracted from the Philadelphia Federal Reserve Bank website. The last line of the table reports the model-implied equivalent correlations; the model is the one presented in Section 2, whose specification is detailed in Table 2.

The low but rather positive (for three subsamples) empirical correlation between GDP growth and expected future GDP growth is not perfectly in line with the present framework. However, it strongly rejects those frameworks featuring a pure auto-regressive consumption growth process, i.e. a model without output gap ($\sigma_z = 0$), for which the correlation between GDP growth and expected GDP growth is equal to one.

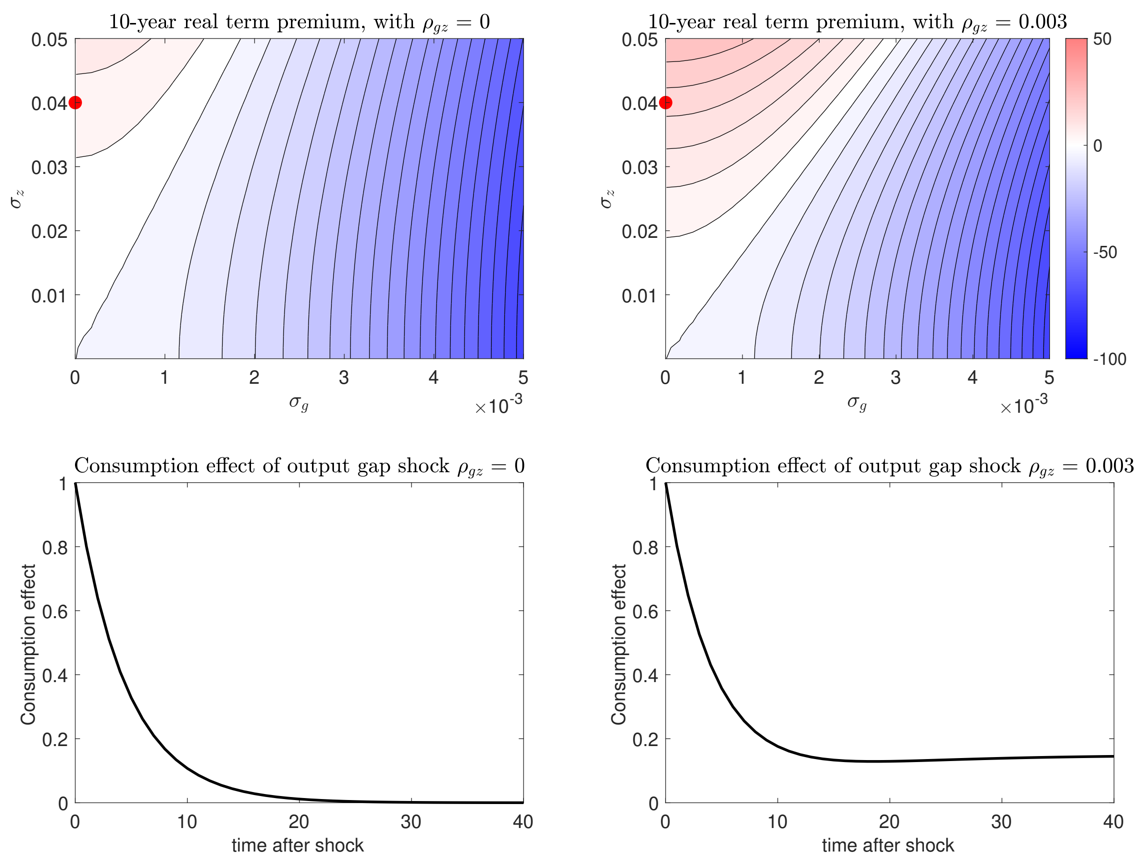
III.4 The relevance of hysteresis effects for the term structure of real rates

Section 2.5 discussed the main model ingredients for generating an upward-sloping real term structure. This annex analyses the amplifying hysteresis channel in more depth.

In our specifications, hysteresis effects are introduced through parameter ρ_{gz} (see eq. 1): if $\rho_{gz} > 0$, periods of negative output gap (z_t) imply reductions in the trend of consumption growth (g_t). Hence, a recession ($z_t < 0$) is a bad state of the world for two compounded reasons: (i) by definition, consumption is low—below it trend—when $z_t < 0$, and (ii) when $\rho_{gz} > 0$, the fact that $z_t < 0$ reduces expected trend growth rate. This is illustrated by the lower plots of Figure III.2, which compare the response of the consumption level (c_t) to increases in z_t in two situations: no hysteresis effect ($\rho_{gz} = 0$) for the left plot and existence of an hysteresis effect ($\rho_{gz} > 0$) for the right plot. The key difference is that while the effect completely dies out when $\rho_{gz} = 0$, it is not the case when $\rho_{gz} > 0$. Consequently, for a given state of recession, hysteresis effects worsen consumption prospects, leading to higher risk prices which, in turn, amplifies forward premiums.

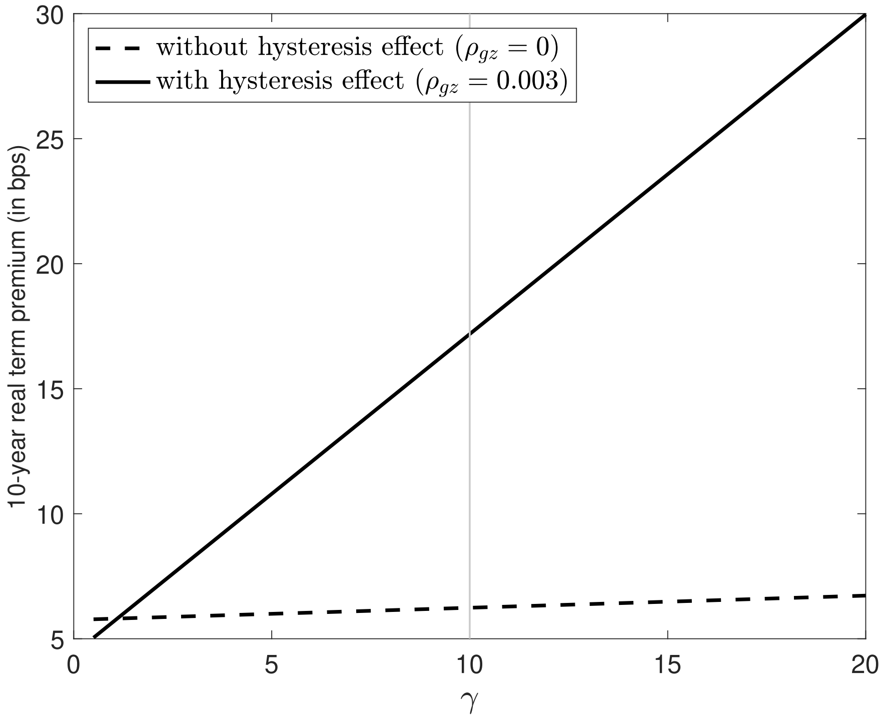
The upper plots of Figure III.2 illustrate the influence of hysteresis effects on real term premiums. Term premiums appear to be larger when $\rho_{gz} > 0$ (i.e., with hysteresis effect, right plot), than when $\rho_{gz} = 0$ (left plot). Figure III.3 shows how the term premiums depend on the coefficient of risk aversion, for the values of σ_z and σ_g indicated on Figure III.2 with red dots. Real term premiums appear to be more sensitive to the coefficient of risk aversion with hysteresis effects.

Figure III.2: Real term premium and hysteresis effect



Notes: This figure illustrates the influence of the permanent and transitory and permanent consumption shocks, as well as the hysteresis effect, on the real term premium. The real term premium is defined in (23)); it is given by $\mathbb{E}(r_{t,40} - r_{t,1})$; it is also the average slope of the term structure of real rates. Only the real part of the model is concerned (i.e., eqs. 1, 2, 12), agents feature Epstein-Zin preferences (see Subsection 2.2), with a constant coefficient of risk aversion. The left plots correspond to the case where $\rho_{gz} = 0$—the situation with no hysteresis effects; by contrast, there is an hysteresis effect in the model underlying the right plots. The upper plots show how the real term premium depends on σ_g and σ_z, that are the respective standard deviations of the shocks affecting the persistent component of consumption *growth* (g_t) and the transitory component of the cyclical component of consumption *level* (z_t). The lower plots show the impulse response functions of (log) consumption c_t to a unit increase in z_t; the bottom-right plot shows that, in a context of hysteresis, these shocks ($\varepsilon_{z,t}$) have a permanent effect on consumption. The upper plots show that real term premium positively depend on σ_z and negatively on σ_g. The top-right plot also shows that, when σ_g is low, the hysteresis effects allow to generate higher (positive) real term premiums. The model parameterization is as follows: $\rho_z = 0.8$, $\rho_g = 0.9$, $\mu_c = 2\%$, $\gamma_t = \mu_{\gamma,0} = 10$. The red dots indicate the values of σ_z and σ_g used in Figure III.3.

Figure III.3: Real term premium, hysteresis effect, and risk aversion



Notes: This figure illustrates the influence of the risk aversion coefficient on the term premium. The real term premium is defined in (23)); it is given by $\mathbb{E}(r_{t,40} - r_{t,1})$; it is also the average slope of the term structure of real rates. We consider two models: one is with hysteresis effects ($\rho_{gz} > 0$, solid line) and the other is without hysteresis effects ($\rho_{gz} = 0$, dotted line). Only the real part of the model is concerned (i.e., eqs. 1, 2, 12), agents feature Epstein-Zin preferences (see Subsection 2.2), with a constant coefficient of risk aversion γ. We consider different values of the coefficient of risk aversion (x axis). The model parameterization is as follows: $\rho_z = 0.8$, $\rho_g = 0.9$, $\mu_c = 2\%$, $\gamma_t = \mu_{\gamma,0} = 10$. The values of σ_z and σ_g are those indicated by red dots in Figure III.2; for the model with hysteresis effects: $\rho_{gz} = 0.003$. The vertical bar indicates the value of the coefficient of risk aversion used in Figure III.2.