Time-varying risk aversion and inflation-consumption correlation in an equilibrium term structure model *

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Abstract

Inflation risk premiums tend to be positive in an economy mainly hit by supply shocks, and negative if demand shocks dominate. Risk premiums also fluctuate with risk aversion. We shed light on this nexus in a linear-quadratic equilibrium macro-finance model featuring time variation in inflation-consumption correlation and risk aversion. We obtain analytical solutions for real and nominal yield curves and for risk premiums. While changes in the inflation-consumption correlation drive nominal yields, changes in risk aversion drive real yields and act as amplifier on nominal yields. Combining a trend-cycle specification of real consumption with hysteresis effects generates an upward-sloping real yield curve. Estimating the model on US data from 1961 to 2019 confirms substantial time variation in inflation risk premiums: distinctly positive in the earlier part of our sample, especially during the 1980s, and turning negative with the onset of the new millennium.

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1 Introduction

Changes in the shape of the yield curve can stem from various sources: shifting inflation expectations, re-assessments of the overall health of the economy, variation in risk aversion or movements in uncertainty and perceived risk. These factors typically have different effects across maturities, on real versus nominal bond yields, and on average short-term rate expectations versus risk premiums. While traditional reduced-form models of the term structure of interest rates lack the ability to distinguish between these drivers and their effects, equilibrium term structure models have the capacity to shed light on the underlying economic mechanisms and on their quantitative relevance. As a result, they have gained heightened attention and momentum over the last decade. Following the definition of Piazzesi and Schneider (2007), equilibrium term structure models feature risk-averse agents who price bonds in the face of stochastic consumption and inflation. These models can be viewed as compromises between purely structural models, such as that by Hördahl et al. (2006) and Dew-Becker (2014), and reduced-form models, exemplified by Ang and Piazzesi (2003), Christensen et al. (2011), or Joslin et al. (2014).

This paper contributes to this literature with a new equilibrium model based on a representative agent with Epstein and Zin (1989) recursive preferences and (exogenous) factor processes governing consumption growth and inflation. We use it to price the term structure nominal bond yields, real rates and inflation compensation.

The model introduces three innovations. First, unlike the bulk of the literature using Epstein-Zin preferences our specification comes with time-varying risk aversion. This additional factor helps generate time variation in real and nominal term premiums.

Second, the model contains a new mechanism that allows for a changing correlation between consumption growth and inflation. It can thereby generate protracted periods during which the economy is predominantly hit by supply shocks and others that are driven by demand shocks. In turn, these changes in the correlation between inflation and consumption growth can become important determinants of the sign and size of inflation risk premiums and nominal term premiums.

Third, the model is able to generate a real yield curve that is upward-sloping on average – a pattern that is observed in the data but that most existing equilibrium models fail to provide (see, e.g., Kung, 2015).[1] At the technical level, an upward sloping real yield curve is only possible if the real stochastic discount factor is negatively auto-correlated (Backus et al., 1989). Yet, under standard preference settings such as power utility or Epstein-Zin, real rates are usually positively influenced by expected consumption growth which, in turn, typically has a strongly positive auto-correlation in these models. As a result, real term premiums and thus the real yield curve slope downwards on average in such a setup. Our solution to generating positive real term premiums on average is simple; it consists in introducing a cyclical component—or output gap—in the consumption *level* process. In that case, expected consumption growth can be positive in a context of a negative output gap. Moreover, as this channel alone yields only minimal (yet positive) real term premiums, we also allow for hysteresis effects whereby cyclical consumption shocks can exert enduring impacts on real growth;[2] this amplifies the magnitude of these premiums.

Despite these innovations, the model allows for analytical pricing formulas for nominal rate, real rates—and thus inflation compensation—at any maturity. Specifically, the introduction of time-varying consumption-inflation correlation renders the model linear-quadratic (rather than purely linear). The tractability is one of the key differences to Boudoukh (1993) who studies the inflation-output correlation in a simpler equilibrium model of nominal bond prices without being able to derive closed-form solutions for bond pricing.

The model is estimated on US quarterly data from 1961 to 2019, including consumption growth, inflation, nominal bond yields, inflation-linked bond yields as well as survey information on future interest rates and inflation. The overall fit is remarkably good in view of the diversity of data from macro, financial and survey sources and the relatively simple and parsimonious structure of the model, which at the same time imposes strong constraints on the joint dynamics of

[1] In the context of Dynamic Stochastic General Equilibrium models, Kısacıkoğlu (2020) shows that standard New-Keynesian models cannot properly account for term structure of real rates.

[2] The resurgence of interest in such effects, as highlighted by Cerra et al. (2023), stems from the enduring consequences of the global financial crisis on GDP in advanced economies and more recent apprehensions regarding the lasting effects of the COVID-19 shock. In a recent study, Furlanetto et al. (2021) identify demand shocks that can have a permanent effect on output through hysteresis effects. Estimated shocks are found to be quantitatively important in the United States, in particular when the Great Recession is included in the sample.

state variables. Nominal term premiums behave similar to standard results in the literature (e.g., Adrian et al., 2013; Kim and Wright, 2005) and the decomposition into real term and inflation risk premiums looks plausible.

We highlight the following empirical findings. First, the described mechanism for generating positive average real term premiums manifests itself empirically: the discussed hysteresis effect is economically significant and the model-implied unconditional expectation of the slope of the real curve is matched to 0.8 percentage points, mirroring a similar magnitude in the data. Second, risk aversion shows distinct time variation with plausible dynamics: despite being inferred only from macro and bond price information, risk aversion captures several salient bouts of changing risk appetite as suggested by other measures that incorporate stock market information, such as Pflueger et al. (2019) and Bauer et al. (2023). Third, the model generates time variation in the correlation between consumption growth and inflation, signalling a clear dominance of supply shocks in the late 1970s and 1980s, and a demand-dominated pattern since the 1990s. Similar to the findings by Breach et al. (2020), supply-shock dominated phases coincide with positive and a demand-shock environment with negative inflation risk premiums. Fourth, risk aversion drives nominal term premiums by creating real term premiums and, importantly, also by amplifying inflation risk premiums.

Related literature

The paper contributes to the literature on equilibrium term-structure models. We feature time-varying risk aversion, which Brandt and Wang (2003) and Dew-Becker (2014) demonstrate to be important for a structural model to satisfyingly fit the yield curve.[3] At the technical level we show that, (a) under Epstein-Zin preferences, (b) with a unit elasticity of intertemporal substitu-

[3] Dew-Becker (2014) investigates the importance of time-varying risk aversion to account for the dynamics of nominal yields in the context of a structural New Keynesian model; his model, which does not entails an analytical solution, is solved numerically. See also Campbell and Cochrane (1999), Gordon and St-Amour (2000), or Melino and Yang (2003), for discussions regarding the importance of time varying risk aversion for asset pricing in general. Bekaert and Engstrom (2017), Bekaert et al. (2021), and Bekaert et al. (2022) develop affine asset pricing models with external habit formation (in the spirit of Campbell and Cochrane, 1999) where risk aversion is time-varying; they do not consider the ability of their framework to jointly account for the term structures of nominal and real rates.

tion, and (c) if the risk aversion parameter is part of a state vector that follows a Gaussian vector auto-regressive process, then the stochastic discount factor (s.d.f.) admits a closed-form solution. Interestingly, the form of the resulting s.d.f. corresponds to that of standard reduced-form Gaussian term structure models, where the vector of prices of risk is affine in the state vector (e.g., Cochrane and Piazzesi, 2005; Piazzesi, 2010; Joslin et al., 2011). To our knowledge, the present paper is the first to provide a derivation of the standard reduced-form s.d.f. from a structural model with Epstein-Zin preferences. A related work is that of Creal and Wu (2020) who allow for time variation in the rate of time preference in the context of Epstein-Zin utility (as in Albuquerque et al., 2016; Schorfheide et al., 2018), and explore the implication of this feature on the term structure of nominal rates. In line with Dew-Becker (2014), we find that while fluctuations in risk aversion are important drivers of yields, there is no significant relationship between risk aversion and the real economy.

As regards the slope of the real curve, in most equilibrium models, the negative correlation between real interest rates and the marginal utility of consumption results from a combination of positively auto-correlated processes giving rise to a negative average slope (e.g., Bansal and Yaron, 2004; Piazzesi and Schneider, 2007; Le et al., 2010; Bansal and Shaliastovich, 2013; Bekaert and Engstrom, 2017; Schorfheide et al., 2018). In order to generate an upward sloping real yield curve Wachter (2006) evokes the external consumption habit framework following Campbell and Cochrane (1999) and shows that it can result in a positive average slope of the term structure of real rates, yet at the cost of introducing a negative relationship between real rates and surplus consumption. Hsu et al. (2021) show that the upward sloping real curve in these models is due to their ability generate a negative auto-correlation of habit-adjusted consumption growth for a sufficiently large habit parameter. Somewhat related, Katagiri (2022) introduces positive real term premiums by creating a negative relationship between real rates and cyclical income. In another approach of Zhao (2020), an ambiguity-averse agent faces different amounts of Knightian uncertainty in the long run versus the short run, making the term structure of expected real rates upward sloping (rather than introducing positive real term premiums).

In terms of modelling approach, our paper shares similarities with the literature on time-varying correlation between nominal bond yields and stock returns. Campbell et al. (2017) study the nominal term structure of yields in a reduced-form linear-quadratic model, while Song (2017) uses a consumption-based equilibrium model for that purpose.

Our paper also relates to the literature on the joint modeling of nominal and real yield curves. Using a no-arbitrage model featuring switching regimes to decompose nominal yields into real and inflation components, Evans (2003) extracts inflation expectations form nominal yields. Ang et al. (2008) employ the same type of models to study the term structure of real rates in the United States. Chernov and Mueller (2012) incorporate survey-based forecasts in a reduced-form Gaussian term-structure model allowing for differences between risk-neutral, subjective, and objective probability measures. Hördahl and Tristani (2012) combine a linear forward-looking macro model and an essentially affine stochastic discount factor to model the nominal and real yield curves and estimate the inflation risk premium. As in the present paper, Breach et al. (2020) rely on quadratic Gaussian term-structure model to explore term premiums. They demonstrate the ability of such a framework to capture diverse macroeconomic dynamics of inflation and real risk premiums, and generate sensible estimates of expected inflation and real short rates over a long sample. Our study demonstrates that these characteristics persist even in a more constrained setting, where the stochastic discount factor is not in reduced form but stems from explicit risk preferences, and when the set of observed variables also contains real variables (e.g., consumption).

The remainder of this paper is organized as follows: Section 2 outlines our modeling framework, including an discussion of the model's ability to capture positive real term premiums; Section 3 presents the estimation approach; Section 4 discusses the empirical results; and Section 5 concludes. The appendix provides a technical description of the general econometric set up; it highlights the dynamic properties of the state variables and provides the pricing formulas. Proofs and additional results are gathered in the supplemental appendix.

2 Model

The model economy is a semi-structural macro-finance model in which a representative agent is faced with exogenous streams of consumption growth and inflation and prices nominal and real bonds in line with Epstein-Zin preferences. The well-known advantage of these preferences are their ability to separate the elasticity of inter-temporal substitution from risk aversion, which helps fit both macro and financial data. Time is indexed with t and refers to quarters.

The model shares many similarities with standard consumption-based asset pricing models, such as closed-form pricing formulas for yields, but also features two distinctive novelties. First, risk aversion is time-varying so as to make term premiums time-varying too. Second, the conditional time-t correlation between consumption growth and inflation is made state-dependant and thus becomes—in addition to risk aversion—another driver of time-varying inflation risk premiums. Both risk aversion and the inflation-consumption correlation are modelled via latent factors.

2.1 The joint dynamics of consumption and inflation

We consider a situation where consumption admits both a permanent (C_t^*) and a transitory zero-mean component (z_t):

$$C_t = C_t^* \exp(z_t).$$

Since $\log(C_t/C_t^*) = z_t$, it comes that z_t can be interpreted as an output gap measure. The average log-growth rate of C_t^* is $\mu_{c,0}$, and g_t denotes the time-varying component of the permament consumption component, that is:

$$\log(C_t^*/C_{t-1}^*) = \mu_{c,0} + g_t.$$

What precedes implies that the (log) consumption growth is:

$$\Delta c_t = \mu_{c,0} + g_t + z_t - z_{t-1},$$

with trend and cyclical components

$$\begin{aligned} g_t &= \rho_g g_{t-1} + \rho_{gz} z_{t-1} + \sigma_g \varepsilon_{g,t} && (1) \\ z_t &= \rho_z z_{t-1} + \sigma_z \varepsilon_{z,t}, && (2) \end{aligned}$$

where $\varepsilon_{g,t}$ and $\varepsilon_{z,t}$ are identically and serially uncorrelated standard normal shocks, and ρ_{gz} controls the hysteresis effect of the cyclical component on trend growth. The shock $\varepsilon_{z,t}$ will also affect the cyclical component of inflation, but in a heteroskedastic way. To obtain that, we posit that $\varepsilon_{z,t}$ is a mixture of two i.i.d. standard normal shocks, namely $\varepsilon_{z,1,t}$ and $\varepsilon_{z,2,t}$:

$$\varepsilon_{z,t} = \frac{1}{\sqrt{2}}(\varepsilon_{z,1,t} + \varepsilon_{z,2,t}), \qquad (3)$$

which is consistent with $\varepsilon_{z,t} \sim i.i.d.\mathcal{N}(0,1)$. While $\varepsilon_{z,1,t}$ and $\varepsilon_{z,2,t}$ have the same (positive) effect on the output gap z_t, they have different effects on inflation. We denote by π_t the continuously-compounded inflation rate between dates $t-1$ and t. Inflation is assumed to be the sum of two stochastic components:

$$\pi_t = \mu_{\pi,0} + \pi_t^* + \tilde{\pi}_t. \qquad (4)$$

While π_t^* and $\varepsilon_{\pi,t}$ are exogenous—in the sense that they do not correlate to consumption growth—$\tilde{\pi}_t$ is conditionally correlated to consumption growth as it is also affected by the two output-gap shocks $\varepsilon_{z,1,t}$ and $\varepsilon_{z,2,t}$ (underlying equations 2 and 3). Specifically:

$$\begin{aligned} \pi_t^* &= \rho_{\pi^*} \pi_{t-1}^* + \sigma_{\pi^*} \varepsilon_{\pi^*,t}, \quad \varepsilon_{\pi^*,t} \sim i.i.d.\mathcal{N}(0,1), && (5) \\ \widetilde{\pi}_t &= \rho_{\tilde{\pi}} \widetilde{\pi}_{t-1} + \sigma_{\pi,z} \left(\frac{1+\kappa_{t-1}}{2} \varepsilon_{z,1,t} - \frac{1-\kappa_{t-1}}{2} \varepsilon_{z,2,t} \right). && (6) \end{aligned}$$

Hence $\tilde{\pi}_t$ is conditionally correlated to the output gap.[4] Importantly, the sign of this conditional correlation depends on that of κ_t. In fact, the one-step-ahead conditional covariance between

[4] If $-1 < \kappa < 1$, then $\varepsilon_{z,1,t}$ and $\varepsilon_{z,2,t}$ have effects of opposite signs on inflation.

inflation and consumption growth and that between the cyclical components is proportional to κ_t:

$$\mathbb{C}ov_t(\pi_{t+1}, \Delta c_{t+1}) = \mathbb{C}ov_t(\widetilde{\pi}_{t+1}, z_{t+1}) = \frac{\sigma_z \sigma_{\pi,z}}{\sqrt{2}} \kappa_t. \tag{7}$$

As a result, the correlation of the cyclical parts boils down to $\mathbb{C}orr_t(\tilde{\pi}_{t+1}, z_{t+1}) = \frac{\kappa_t}{\sqrt{1+\kappa_t^2}}$. Intuitively, factor κ_t determines whether the economy is dominantly driven by demand shocks ($\kappa_t > 0$) or supply shocks ($\kappa_t < 0$). Factor κ_t follows an auto-regressive process of order one:[5]

$$\kappa_t = \mu_\kappa + k_t, \quad \text{and} \quad k_t = \rho_k k_{t-1} + \sigma_k \varepsilon_{k,t}. \tag{8}$$

The unconditional mean μ_κ of that process influences the magnitude and sets the sign of the unconditional correlation between inflation and consumption growth.

2.2 Agents' preferences

The preferences of the representative agent are of the Epstein and Zin (1989) type, with a unit elasticity of intertemporal substitution (EIS). Using a unit EIS facilitates the resolution, i.e., the computation of the stochastic discount factor with closed form solutions (e.g., Piazzesi and Schneider, 2007; Seo and Wachter, 2018, among others).

Specifically, the time-t log utility of a consumption stream (C_t) is recursively defined by[6]

$$u_t = \log U_t = (1-\delta)c_t + \frac{\delta}{1-\gamma_t} \log\left(\mathbb{E}_t \exp\left[(1-\gamma_t)u_{t+1}\right]\right), \tag{9}$$

where c_t denotes the logarithm of the agent's consumption level C_t, δ the pure time discount factor and γ_t is the risk aversion parameter. The latter is assumed to be time-varying:

$$\gamma_t = \mu_{\gamma,0} + w_t + m_t, \tag{10}$$

[5] Equations (5) to (8) also imply that the inflation process is heteroskedastic with $\mathbb{V}ar_t(\pi_{t+1}) = \sigma_{\pi^*}^2 + (1+\kappa_t^2)\frac{\sigma_{\pi,z}^2}{2}$.

[6] Eq. (9) is the limit of the general Epstein and Zin (1989) recursive utility as the EIS approaches one.

with

$$w_t = \rho_w w_{t-1} + \sigma_w \varepsilon_{w,t} \quad \text{and} \quad m_t = \rho_m m_{t-1} + \sigma_m \varepsilon_{m,t}. \tag{11}$$

The two factors w_t and m_t are exogenous sources of variation in γ_t and serve as fast- and slow-moving components in risk aversion. This distinction will become important for the pricing of longer-term bonds.

2.3 Gaussian linear-quadratic state-space representation of the model

The model outlined above can be cast into a quadratic state-space representation, a formulation that will prove useful for pricing and estimation purpose.

First, consumption growth and risk aversion are affine in $X_t = [g_t, z_t, z_{t-1}, w_t, m_t, k_t]'$:

$$\Delta c_t = \mu_{c,0} + \underbrace{\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}}_{\mu'_{c,1}} X_t \tag{12}$$

$$\gamma_t = \mu_{\gamma,0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}}_{\mu'_{\gamma,1}} X_t. \tag{13}$$

Further, we have

$$\pi_t = \mu_{\pi,0} + \mu'_{\pi,Z} Z_t, \tag{14}$$

with $Z_t = [\pi_t^* \quad \widetilde{\pi}_t]'$, and $\mu_{\pi,Z} = [1 \quad 1]'$.

The joint dynamics of X_t and Z_t can be represented as follows:

$$\begin{bmatrix} X_t \\ Z_t \end{bmatrix} = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi_Z \end{bmatrix} \begin{bmatrix} X_{t-1} \\ Z_{t-1} \end{bmatrix} + \begin{bmatrix} \Sigma \\ \Sigma_Z(X_{t-1}) \end{bmatrix} \varepsilon_t, \quad \varepsilon_t \sim i.i.d.\mathcal{N}(0, I_{n_\varepsilon}), \tag{15}$$

where $\varepsilon_t = [\varepsilon_{g,t}, \varepsilon_{z,1,t}, \varepsilon_{z,2,t}, \varepsilon_{w,t}, \varepsilon_{m,t}, \varepsilon_{k,t}, \varepsilon_{\pi^*,t}]'$, and where $\Sigma_Z(X)$ linearly depends on X, which means it satisfies:

$$vec(\Sigma_Z(X_t)) = \Gamma_0 + \Gamma_1 X_t. \tag{16}$$

Appendix A provides the expanded expressions of Φ, Σ, Φ_Z, Γ_0 and Γ_1.

Importantly, the dynamics of $[X_t', Z_t']'$ is of the Gaussian linear-quadratic type, which makes it particularly tractable (e.g., Leippold and Wu, 2002; Ang et al., 2011; Kim and Singleton, 2012; Breach et al., 2020). More formally, the augmented state vector $Y_t = [X_t', Z_t', vech(X_t X_t')]'$ is an affine process, in the sense that its one-period-ahead conditional Laplace transform is exponential affine in its current value (see, e.g., Duffie et al., 2002). That is:

$$\mathbb{E}_t(\exp(u'Y_{t+1})) = \exp\left(\psi_{Y,0}(u) + \psi_{Y,1}(u)'Y_t\right), \tag{17}$$

where the functions $\psi_{Y,0}$ and $\psi_{Y,1}$ admit simple closed-form solutions (Proposition 2 in Appendix C). The affine property of the augmented state-vector ensures that multi-horizon Laplace transforms can be computed in a fast way, which is key to price long-dated financial instruments.

2.4 Stochastic discount factor, risk-neutral dynamics, and bond prices

A key implication of the model outlined above is that the stochastic discount factor (s.d.f.) admits an exponential affine expression. More generally, as demonstrated in Appendix C (Proposition 1), when (i) agents feature Epstein-Zin preferences with unit EIS, (ii) when both the consumption growth rate and the risk aversion parameter are affine functions of a factor X_t (as in eqs. 12 and 13), and (iii) if X_t follows a Gaussian vector auto-regressive process (as in eq. 15), then the (real) s.d.f. admits the exponential affine expression:

$$\mathscr{M}_{t,t+1} = \delta\left(\frac{C_{t+1}}{C_t}\right)^{-1}\frac{U_{t+1}^{1-\gamma_t}}{\mathbb{E}_t(U_{t+1}^{1-\gamma_t})} \tag{18}$$

$$\Leftrightarrow \mathscr{M}_{t,t+1} = \exp\left[-(\eta_0 + \eta_1{}'X_t) + \lambda_t'X_{t+1} - \lambda_t'\Phi X_t - \frac{1}{2}\lambda_t'\Sigma\Sigma'\lambda_t\right], \tag{19}$$

where λ_t, the vector of prices of risk, is affine in X_t, i.e., $\lambda_t = \lambda_0 + \lambda_1' X_t$. The expressions of η_0, η_1, λ_0, and λ_1 are given in Proposition 1. The short-term real rate r_t is affine in X_t:

$$r_t = -\log[\mathbb{E}_t(\mathscr{M}_{t,t+1})] = \eta_0 + \eta_1' X_t. \tag{20}$$

It is worth noting that the s.d.f. expression (19) corresponds to the reduced-form specification underlying the Gaussian Term Structure Models (GTSMs) with affine prices of risk popularized by Ang and Piazzesi (2003) and Kim and Wright (2005). To our knowledge, the present paper is the first to provide a structural interpretation of the reduced-form specification by deriving it from the standard s.d.f. in eq. (18) implied by Epstein-Zin preferences with unit EIS.

Equation (19) characterizes the real s.d.f. The nominal s.d.f., given by $\mathscr{M}_{t,t+1}\exp(-\pi_{t+1})$, is also exponential affine, but in the extended state vector Y_t. Since π_t linearly depends on Y_t, which is an affine process, it comes that the nominal short-term rate is also affine in Y_t. Formally:

$$i_t = -\log[\mathbb{E}_t(\mathscr{M}_{t,t+1}\exp(-\pi_{t+1}))] = \eta_0^{\$} + \eta_1^{\$\prime} Y_t. \tag{21}$$

With the s.d.f. in hand, we can define the risk-neutral dynamics of the state vector. The risk-neutral measure ($\mathbb{Q}$) is obtained, relative to the physical measure ($\mathbb{P}$), by applying the Radon-Nikodym derivative $\mathscr{M}_{t,t+1}/\mathbb{E}_t(\mathscr{M}_{t,t+1})$. Proposition 5 (Appendix C) shows that the conditional risk-neutral Laplace transform of Y_t is of the same form as its physical counterpart. This implies that X_t and Z_t exhibits the same type of dynamics under $\mathbb{Q}$ as under $\mathbb{P}$—albeit with a different parameterization. Importantly, the dynamics of the extended state vector Y_t is also affine under the risk-neutral measure, which opens the door to the affine pricing machinery (Duffie et al., 2002).[7] This is illustrated by the next subsection.

Since the real and nominal short-term rates are affine in Y_t (see eqs. 20 and 21), and because the latter is affine under the risk-neutral measure, it comes that the yields-to-maturity of real and

[7] This is ensured as soon as Y_t is an affine process under $\mathbb{P}$ and that the Radon-Nikodym derivative is exponential affine in Y_t.

nominal zero-coupon bonds of any maturity h are affine in Y_t. Specifically:

$$r_{t,h} = \alpha_h + \beta_h' X_t, \quad \text{and} \quad i_{t,h} = \alpha_h^{\$} + \beta_h^{\$\prime} Y_t, \tag{22}$$

where (α_h, β_h) and $(\alpha_h^{\$}, \beta_h^{\$})$ are given in Propositions 6 and 7, respectively.

2.5 The term structure of real rates

This paper aims to build an equilibrium model that captures the joint dynamics of the real and nominal term structures of interest and makes it possible to decompose real and nominal interest rates into their two components (expectations and term premiums). For that, a necessary condition is for the model to reproduce the average slopes of the term structures of real and nominal yield curves that we observe in the data. The average slopes of the yield curves also correspond to the average term premiums.[8] However, as mentioned in the introduction, the literature on equilibrium term structure models points to the difficulty these models have in accommodating the average upward-sloping term structure of real rates or, equivalently, upward-sloping real term premiums (e.g., Piazzesi and Schneider, 2007; Zhao, 2020; Ellison and Tischbirek, 2021). It is important to note that these difficulties mainly concern real rates, the average slope of nominal rates being easily fixed—in any type of equilibrium model—by adjusting the average covariance between inflation and consumption growth.

Bond term premiums are the components of bond yields that result from agents' risk aversion. For a given maturity (h, say), the term premium is usually defined as the difference between the yield-to-maturity of a maturity-h bond and the expected return of a strategy that involves rolling over one's investment at each period, by placing it in a one-period bond (over the next h periods). If agents were not risk averse, term premiums would be zero. Accordingly, the standard definition

[8] Taking expectations of both sides of (23) gives: $\mathbb{E}(TP_{t,h}) = \mathbb{E}(r_{t,h}) - \mathbb{E}(r_t)$ and $\mathbb{E}(TP_{t,h}^{\$}) = \mathbb{E}(i_{t,h}) - \mathbb{E}(i_t)$.

of the real and nominal term premiums are, respectively (e.g., Gürkaynak and Wright, 2012):

$$TP_{t,h} = r_{t,h} - \mathbb{E}_t\left(\frac{1}{h}(r_t + \cdots + r_{t+h-1})\right), \quad \text{and} \quad TP^{\$}_{t,h} = i_{t,h} - \mathbb{E}_t\left(\frac{1}{h}(i_t + \cdots + i_{t+h-1})\right), \tag{23}$$

which also means that each type of interest rate comprises two components: an expectation part and the term premium.

Real term premiums depend on two ingredients: agents' risk preferences and the consumption process. Let us focus on the second ingredient. In most equilibrium term-structure models, the modeling of the consumption process is generally based on that of the consumption growth rate. Specifically, in the spirit of the long-run risk (LRR) literature initiated by Bansal and Yaron (2004), these studies usually consider that the consumption growth rate is a combination of volatile and autoregressive components. As shown in Online Appendix III, these specifications mechanically lead to negative real term premiums in the context of power-utility time-separable utilities. This online appendix also shows that real term premiums can become positive when we introduce a cyclical component to the consumption level (such as our z_t factor). To get the intuition behind this result, notice that, in equilibrium models, real rates positively depend on expected consumption. As a result, in those specifications where consumption *growth* is positively auto-correlated—as is the case in LRR models—real rates tends to be lower when consumption is low. Hence, the price of (real) bonds tend to be higher in bad states of the world—states of high marginal utilities. Real bonds therefore constitute hedges against bad states of the world, which leads agents to buy them even if their yield-to-maturity are lower than expected future short-term rates; this accounts for the negative real premiums. The mechanism is different when the consumption process—in levels—admits a cyclical component (such as z_t). Indeed, in that case, expected consumption may be negatively correlated to the value of the cyclical component. To be sure, if $c_t = z_t$ and if $z_t = \rho z_{t-1} + \varepsilon_t$ (with $0 < \rho < 1$ and $\mathbb{E}_t(\varepsilon_{t+1}) = 0$), then $\mathbb{E}_t(\Delta c_{t+1}) = (\rho - 1)c_t$. Since $\rho - 1 < 0$, it comes that expected consumption growth is higher when consumption is low. In this situation, a real bond loses value in bad states of the world, leading to a positive term premium.[9]

[9] Using surveys on US GDP, Appendix III.3 points to a low but rather positive correlation between GDP growth and

What precedes indicates that the introduction of cyclical components in the level of consumption can help generating positive real term premiums. It however appears that this results in term premiums of limited size. Hysteresis effects can help amplify real risk premiums. Generally speaking, hysteresis effects are mechanisms through which demand shocks can exert enduring impacts on real output. In our specifications, hysteresis effects are introduced through parameter ρ_{gz} in eq. (1) connecting the output gap to trend consumption growth. Appendix III.4. discusses how this hysteresis channel amplifies the dynamic effect of recessions on both cyclical consumption growth and trend growth, and how it eventually affects real term premiums.

3 Estimation approach

The model is defined in eqs. (1) to (14) and consists of the state vector $X_t = [g_t, z_t, z_{t-1}, w_t, m_t, k_t]'$ that drives real variables, of the extended state vector $Y_t = [X_t', Z_t', vech(X_t X_t')]'$ with $Z_t = [\pi_t^*, \tilde{\pi}_t]'$ for nominal variables, the innovation vector $\varepsilon_t = [\varepsilon_{g,t}, \varepsilon_{z,1,t}, \varepsilon_{z,2,t}, \varepsilon_{w,t}, \varepsilon_{m,t}, \varepsilon_{k,t}, \varepsilon_{\pi^*,t}]'$, and the parameters $\theta = [\rho_g, \rho_z, \rho_w, \rho_m, \rho_k, \sigma_g, \sigma_z, \sigma_w, \sigma_m, \sigma_k, \rho_{\pi^*}, \sigma_{\pi^*}, \rho_{\tilde{\pi}}, \sigma_{\pi,z}, \mu_\pi, \mu_c, \mu_\gamma, \mu_\kappa, \delta, \rho_{g,z}]$.

We estimate the model and these 20 parameters by maximumizing the likelihood function, using quarterly data for the United States from 1961Q2 until 2019Q4. For that purpose, we cast the model into a linear-quadratic state-space representation (see Subsection 2.3 for the state dynamics). The computation of the quasi likelihood function is based on the Quadratic Kalman Filter (QKF) introduced by Monfort et al. (2015), that specifically handles the estimation of Gaussian linear-quadratic models. This filter runs under the assumptions that observed variables are affine in the extended state vector Y_t, which is the case for the observed variables we consider (listed in Table 1). The filter exploits the fact that the dynamics of the extended state vector takes the form of a linear vector auto-regressive process whose conditional covariance is not constant but depends linearly on the lagged state. See Appendix B for additional details.

expected GDP growth. While this evidence is not perfectly in line with the present framework, it strongly rejects those frameworks featuring a pure auto-regressive consumption growth process. Indeed, in the latter case, the correlation between GDP growth and expected GDP growth is equal to one.

As observable variables, we include real per-capita consumption growth, CPI inflation and, for reasons of data availability, a GDP-based output gap for the macroeconomic block. The financial block consists of the three-month Treasury bill rate from FRED (corresponding to one period in our quarterly model) as well as nominal yields and real yields based on TIPS with maturities from 2 to 20 years—taken from the updated Gürkaynak et al. (2007) database. Due to the relatively late existence of TIPS, we augment the dataset to the extent possible with backcasted real rates from Groen and Middeldorp (2013), who estimate them in regressions of TIPS yields on a vast set of over 100 macro and financial variables, using partial least squares to discipline the selection of variables. On top of the pure financial and macroeconomic variables, we also incorporate survey-based forecasts among the observed variables. This approach, popularized by Kim and Wright (2005) and Kim and Orphanides (2012), aims at capturing better the typically high persistence in the estimation of term structure models (e.g., Jardet et al., 2013; Bauer et al., 2012). We include surveys on average inflation (CPI10) and the average short-term interest rate (BILL10) over the next 10 years taken from the Survey of Professional Forecasters of the Federal Reserve Bank of Philadelphia to anchor the average expectations of these variables in the model. While CPI10 is available at a quarterly frequency, we linearly interpolate the slow-moving BILL10 which is only made available once a year. Furthermore, we include survey-data for the perceived inflation target (PTR) and for the expected federal funds rate in the long run (RTR) used in the FRB/US model (Brayton et al., 2014) to anchor the endpoints of expected inflation and the expected short-term nominal rate. We assume the endpoints to be reached in 10 years' time.

Our econometric model comprises 16 measurement equations to match the observed variables listed in Table 1 with their model-implied counterpart plus measurement errors. The feasibility of this approach relies on the existence of (i) analytical pricing formulas (see Appendix D) and (ii) analytical moment formulas (see Online Appendix II). The standard deviation of the measurement error for the output gap measured in percent is set to 0.2 percentage points and for the other 15 variables, which are all growth rates or yields expressed in percent, to 0.1 percentage points. The higher value for the output gap accounts for the fact that it is based on potential output which

Table 1: Observable variables and model counterparts

Type	Variable	Mean	S.D.	Min	Max	First date	Model ×100
Macro	Consumption	0.55	0.65	−2.58	2.49	Jun-1961	Δc_t
	CPI inflation	0.92	0.81	−3.48	4.08	Jun-1961	π_t
	Output gap	−0.86	2.33	−8.16	5.59	Jun-1961	z_t
Nominal	YIELD3M	4.57	3.19	0.01	15.49	Jun-1961	i_t
yield	YIELD02	5.24	3.25	0.22	15.78	Jun-1961	$i_{t,8}$
	YIELD05	5.69	3.02	0.65	15.18	Jun-1961	$i_{t,20}$
	YIELD10	6.33	2.98	1.53	14.89	Sep-1971	$i_{t,40}$
	YIELD20	6.31	2.89	2.01	14.84	Sep-1981	$i_{t,80}$
Real	TIPSY02	0.56	1.58	−2.10	4.22	Mar-1999	$r_{t,8}$
yield	TIPSY05	1.09	1.47	−1.68	4.28	Mar-1999	$r_{t,20}$
	REALR10	2.23	1.06	−0.74	4.29	Sep-1971	$r_{t,40}$
	REALR20	2.51	0.91	0.24	4.24	Sep-1971	$r_{t,80}$
Survey	RTR	5.16	1.64	2.39	9.72	Jun-1961	$\mathbb{E}_t i_{t+40}$
	PTR	3.19	1.56	1.68	7.72	Jun-1961	$\mu_\pi^{PCE} + 4\mathbb{E}_t \pi^*_{t+40}$
	BILL10	3.67	0.89	2.37	5.22	Mar-1992	$\frac{1}{40}\sum_{h=1}^{40} \mathbb{E}_t i_{t+h}$
	CPI10	2.62	0.44	2.14	4.02	Dec-1991	$\frac{4}{40}\sum_{h=1}^{40} \mathbb{E}_t \pi_{t+h}$

Notes: All variables are expressed in percent and growth/inflation rates are quarter-on-quarter. Data runs from the date indicated in the table until 2019Q4. Consumption refers to the real per-capita growth rate based on population, nominal consumption of all goods and services and the related price index from the Bureau of Economic Analysis. CPI inflation refers to all items and is from the Bureau of Labor Statistics. Output gap is log real GDP from the Bureau of Economic Analysis minus log potential GDP from the Congressional Budget Office. The 3-month treasury bill rate (YIELD3M) is from FRED, other nominal (YIELDX) and real (TIPSX) interest rates from the updated Gürkaynak et al. (2007) database and backcasted real rates (REALRX) from Groen and Middeldorp (2013). The perceived inflation target (PTR) and the expected federal funds rate in the long run (RTR) are from the FRB/US model (Brayton et al., 2014). Survey of Professional Forecasters (SPF) data (mean of forecasts) are from the Federal Reserve Bank of Philadelphia for averages over the next ten years of CPI inflation (CPI10) and of the 3-month treasury bill rate (BILL10).

is an estimated and not an observed variable itself. Likewise, real yields before 1999 and RTR before 1984 are based on constructed and not observable data and thus the standard deviations of their measurement errors are also doubled in those periods. Finally, due to massive outliers in quarter-on-quarter inflation and associated liquidity premiums TIPS yields in Q3 and Q4 2008 (see Fleckenstein et al., 2014; D'Amico et al., 2018) the standard deviations of the measurement errors are increased tenfold for those individual observations.

Whereas PTR refers to PCE inflation, the other inflation rates refer to the CPI. The measurement equation for PTR, which is matched to trend inflation, takes into account that CPI inflation was on average 44 basis points higher in annualised terms than PCE inflation in our sample.[10]

[10] In the notation of the quarterly model, the spread between PCE and CPI inflation equals $\mu_\pi^{PCE} = 4\mu_\pi - 0.44/100$.

While an unconstrained estimation of the 20 parameters listed in Table 2 results in a satisfactory fit of the observed data, it fails to provide satisfactory unconditional moments. Hence, we freely estimate all parameters except μ_c, μ_γ, σ_m and σ_k which are estimated via nested constraints. Specifically, conditional on the other parameters in the estimation, the parameters μ_c and μ_γ are chosen such that the model-implied unconditional real rates at the 2 and 10 year maturity match 1.5 and 2.3 percent respectively. While the value for the 10-year maturity is based on our sample average, the sample average for the 2-year maturity appears too low as a result of its shorter sample that includes the zero lower bound era. Our matched real rates imply a positive slope of the real yield curve of 0.8 percentage points, in line with the literature using US data (see, e.g., Swanson, 2015; Zhao, 2020; Hsu et al., 2021). The parameter σ_k is solved for such that the factor κ_t remains in the range between -1 and +1 with a very high probability of 99.9 percent. The discussion around equation 8 clarifies the meaning of that range. Finally, the parameter σ_m is set to a value such that the unconditional standard deviation of period-on-period changes in the slow-moving risk aversion factor m_t equals 1. The latter constraint is based on changes rather than levels as the factor m_t is highly persistent and thus its unconditional variance in levels is less meaningful.

4 Empirical results

4.1 Parameter estimates, fit and diagnostics

The resulting parameter values are summarized alongside their estimated standard deviations in Table 2. All parameters are statistically significant at the 90% confidence level. The auto-regressive parameters $\rho_\bullet$ of the macro processes are highly persistent, including both drivers (w and m) of risk aversion. The unconditional mean of risk aversion, μ_γ is close to 24. In the literature, estimated values for the risk aversion parameter in asset pricing models with Epstein-Zin preferences vary widely from low single digit to high double digit numbers (see, e.g., Bansal et al., 2007; Rudebusch and Swanson, 2012; Bansal and Shaliastovich, 2013; Chen et al., 2013; Creal and Wu, 2020). The point estimate of the hysteresis parameter $\rho_{g,z}$, capturing the impact of the consumption/output gap on the growth rate of permanent consumption, see eq. (1), looks small at first glance but it is economically significant: setting the parameter to zero—ceteris paribus—leads to a marked increase in the loss function as it makes matching the positive slope of the unconditional real yield curve much more costly for the other parameters. The unconditional mean μ_κ of the factor κ, controlling the time-varying consumption-inflation correlation, is negative which implies that the economy in steady state is in a supply-shock driven environment, which is in line with an average upward sloping nominal term structure. The empirical estimate of the yield curve is discussed in more detail in Subsection 4.2.

Figure 1 shows the fit of observed variables together with the mean absolute fitting error next to each variable's name. To produce model-implied variables, we use the filtered estimates of X_t (and thus of X_tX_t') and Z_t resulting from the QKF. The overall fit is remarkably good in view of the diversity of data from macro, financial and survey sources and the relatively simple and parsimonious structure of the model, which at the same time imposes strong constraints on the joint dynamics of state variables.

The macro variables (real consumption growth, inflation, and the output gap) are fitted well. Figure 2 shows that the inflation is driven by fluctuations of the cyclical component, $\tilde{\pi}$, around the trend component, π^*, which is moving slowly but considerably within the sample.

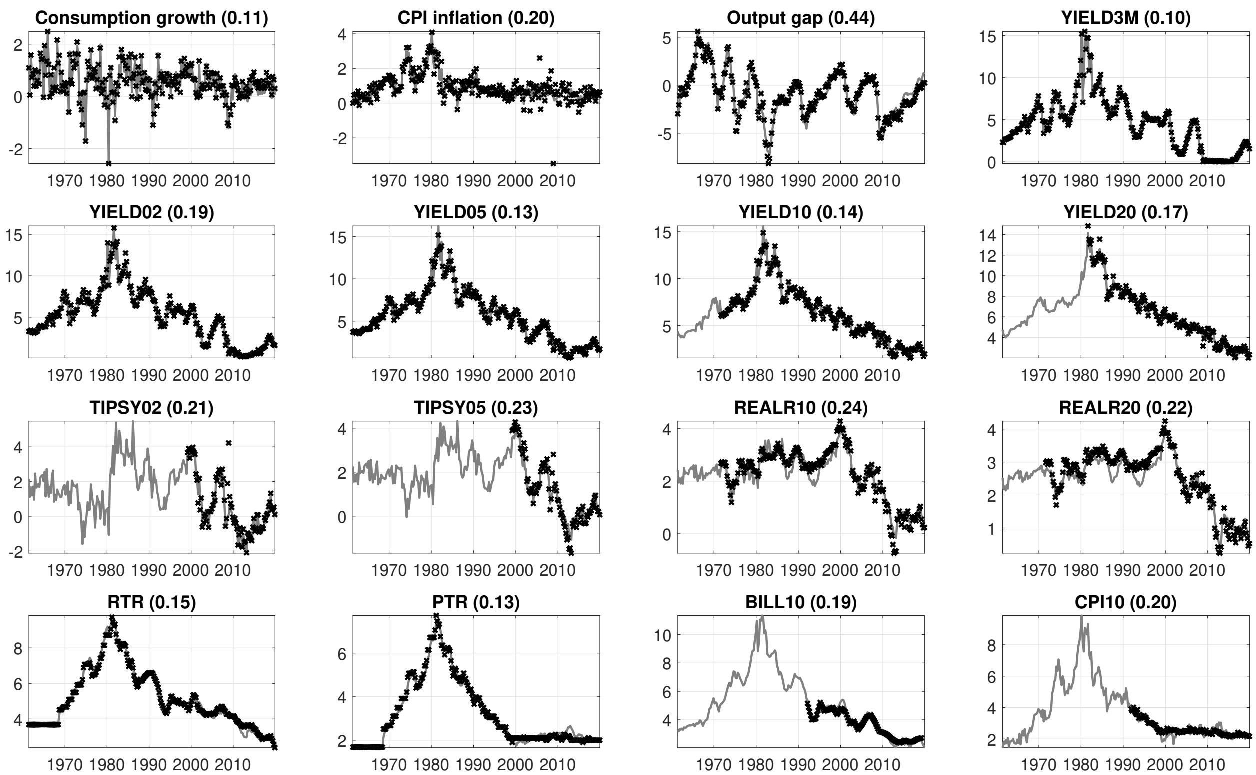
Nominal bond yields have a mean absolute fitting error ranging from 12 to 21 basis points across maturities. For real rates the mean absolute error ranges more narrowly from 20 to 23 basis points along the term structure. Overall, these pricing errors are not fully comparable to the typically smaller fitting errors of latent-factor models operating purely on yield information, in particular for nominal yields. These reduced-form models have greater flexibility in filtering latent factors, whereas our model variables—including bond yield drivers—operate under the rich parameter constraints imposed by our preference structure and macro dynamics. Regarding surveys, the model manages to capture the level and dynamics of our survey variables reasonably well.

Table 2: Parameter estimation

Description	Parameter	Point estimate	Std. Dev.	Confidence range	
AR trend growth	ρ_g	0.99092	0.00012	0.99073	0.99111
AR consumption gap	ρ_z	0.96729	0.00034	0.96674	0.96785
AR fast risk aversion	ρ_w	0.93298	0.00068	0.93185	0.93410
AR slow risk aversion	ρ_m	0.98397	0.00050	0.98314	0.98479
AR $corr(\pi, c)$ factor	ρ_k	0.92495	0.00095	0.92339	0.92652
SD trend growth	σ_g	0.00020	0.00000	0.00020	0.00021
SD output gap	σ_z	0.03309	0.00035	0.03251	0.03366
SD fast risk aversion	σ_w	4.78928	0.12514	4.58255	4.99601
SD slow risk aversion	σ_m	0.99598	0.00013	0.99577	0.99619
SD $corr(\pi, c)$ factor	σ_k	0.07873	0.00096	0.07715	0.08030
AR trend inflation	ρ_{π^*}	0.99371	0.00021	0.99336	0.99406
SD trend inflation	σ_{π^*}	0.00026	0.00000	0.00025	0.00027
AR cyclical inflation	$\rho_{\tilde{\pi}}$	0.91981	0.00071	0.91863	0.92098
SD cyclical inflation	$\sigma_{\pi,z}$	0.01577	0.00016	0.01551	0.01603
mean inflation	μ_π	0.00892	0.00017	0.00863	0.00920
mean consumption growth	μ_c	0.00588	0.00004	0.00581	0.00594
mean risk aversion	μ_γ	24.01223	0.21167	23.66256	24.36191
mean $corr(\pi, c)$ factor	μ_κ	−0.35991	0.00462	−0.36754	−0.35229
time discount factor	δ	0.99362	0.00005	0.99354	0.99370
hysteresis effect	$\rho_{g,z}$	0.00011	0.00000	0.00010	0.00012

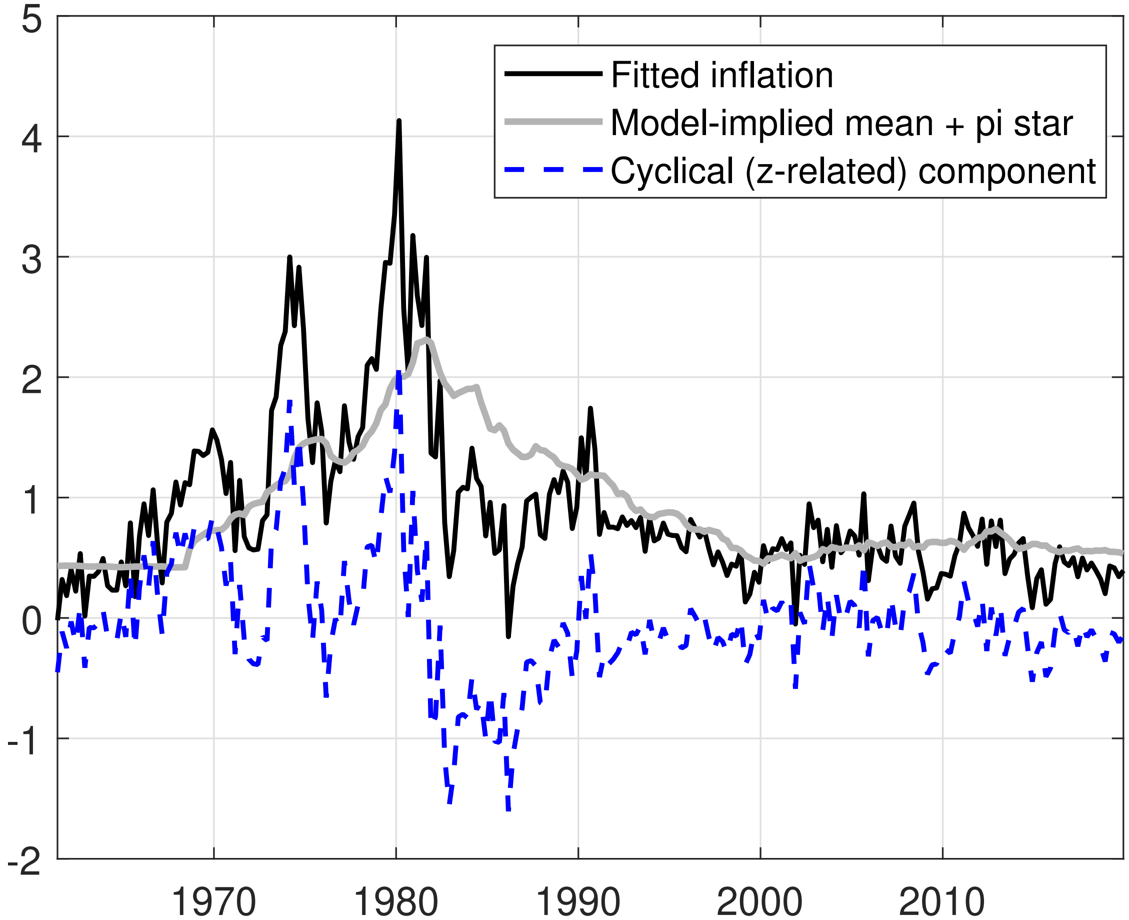
Notes: This table shows the model estimates from the Maximum Likelihood Estimation of our state-space model using a Quadratic Kalman Filter. The covariance matrix of the parameters is obtained via the outer product of gradients. Conditional on the other parameters in the estimation, μ_c, μ_γ, σ_m and σ_k are solved to match certain unconditional moments for real rates, the risk aversion process and the κ process driving the inflation-consumption correlation. Standard deviations for these four parameters are obtained by drawing 10,000 times from the multivariate normal distribution of all other parameters and their estimated covariance matrix and then solving for μ_c, μ_γ, σ_m and σ_k subject to their constraints. See Subsection 3 for details. The confidence range refers to the 90% confidence interval.

Figure 1: Model fit



Notes: This figure compares observed variables with model-implied ones. The model-implied variables depend on the estimates of the latent factors, i.e., the components of X_t and Z_t (see Subsection 2.3; these estimates result from the Quadratic Kalman Filter (Monfort et al., 2015). The numbers next to each variable's name indicate the mean absolute fitting error for observable data points. All variables and errors are expressed in percent.

Figure 2: Inflation decomposition



Notes: This figure shows the decomposition of inflation (π_t) into its two components (see eq. 4). The estimated components are estimated by the Quadratic Kalman Filter.

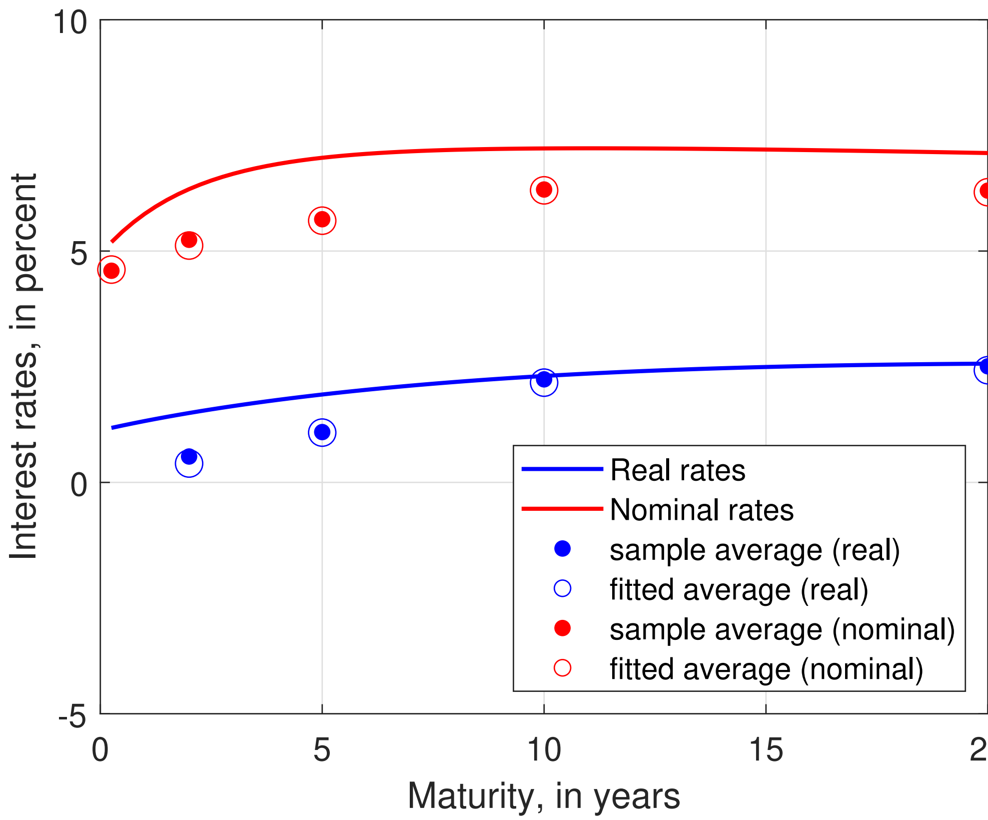
4.2 Model-implied unconditional and fitted premiums

The model-implied unconditional mean of the nominal term structure is upward-sloping and stands somewhat above the sample average of the data (see Figure 3). The latter result stems, among other things, from the negative estimate of μ_κ which implies a negative unconditional correlation between inflation and consumption growth and thus positive unconditional inflation risk premiums that increase with maturity (see also Table 3). As the inflation-consumption correlation is allowed to vary over time, this feature of the unconditional nominal term structure is without larger implications for fitting nominal yields over time (see dots and circles for comparison in Figure 3).

The model-implied unconditional expectation of the term structure of real rates is likewise upward-sloping, which in turn—as we have a stationary model—is due to real term premiums that are positive on average and increase with maturity. The structural model ingredients for that feature to arise have been discussed in Subsection 2.5 above: a cyclical component affecting the level of real consumption, the presence of hysteresis by which cyclical shocks have a permanent effect on trend consumption growth, and a non-trivial nexus between risk aversion and real term premiums. While several of the (semi-)structural models in the literature imply an average real term structure that is downward-sloping or only mildly upward-sloping, our estimation obtains a match of the real term structure with reasonable parameters.

Table 3 shows that the volatility of nominal term premiums is halved when when switching off the volatility of risk aversion and it is reduced roughly to one fifth when setting the inflation-consumption correlation to zero. But keeping the inflation-consumption correlation constant at its baseline value results in a much lower reduction of volatility of nominal term premiums. This finding is in line with Creal and Wu (2020) who identify time variation in risk pricing as the main driver of bond term premiums rather than stochastic volatility. The dependence of nominal term premiums on risk aversion and inflation-consumption correlation reflects a combination of how real term premiums and inflation risk premiums depend on these factors. For real term premiums, the results confirm that risk aversion is—by construction of our model—the only source of time variation. By contrast, the variation of inflation risk premiums depends on both variations

Figure 3: Unconditional term structures of interest rates



Notes: This figure shows (model-implied) unconditional term structures of real and nominal interest rates and the average of observed and fitted rates over the available sample.

in risk aversion and the inflation-consumption correlation to a similar degree.[11] Moreover, inflation risk premiums turn constant if the inflation-consumption correlation is zero. In other words, risk aversion serves as amplifier of inflation risk premiums and therefore operates only if these are non-constant due to a non-zero correlation between inflation and consumption growth.

Looking at the estimated premiums over time, the 5-year and 10-year nominal term premiums shown in Figure 4 display reasonable magnitudes and exhibit similar dynamics as two prominent candidate estimates in the literature by Kim and Wright (2005) (KW) and Adrian et al. (2013) (ACM). For the sample since 1990, where all three measures are available, our term premiums are close to those of KW noting that both approaches rely on survey information, while ACM does not. Compared to the 'pure-yields' model results by ACM that can be compared over the full sample,

[11] In linear models, a historical or a forecast-error variance decomposition would be used to quantify the contributions of different shocks. These decompositions are not possible in our linear-quadratic setup.

Table 3: Effects of time-varying risk aversion and inflation-consumption correlation on premiums

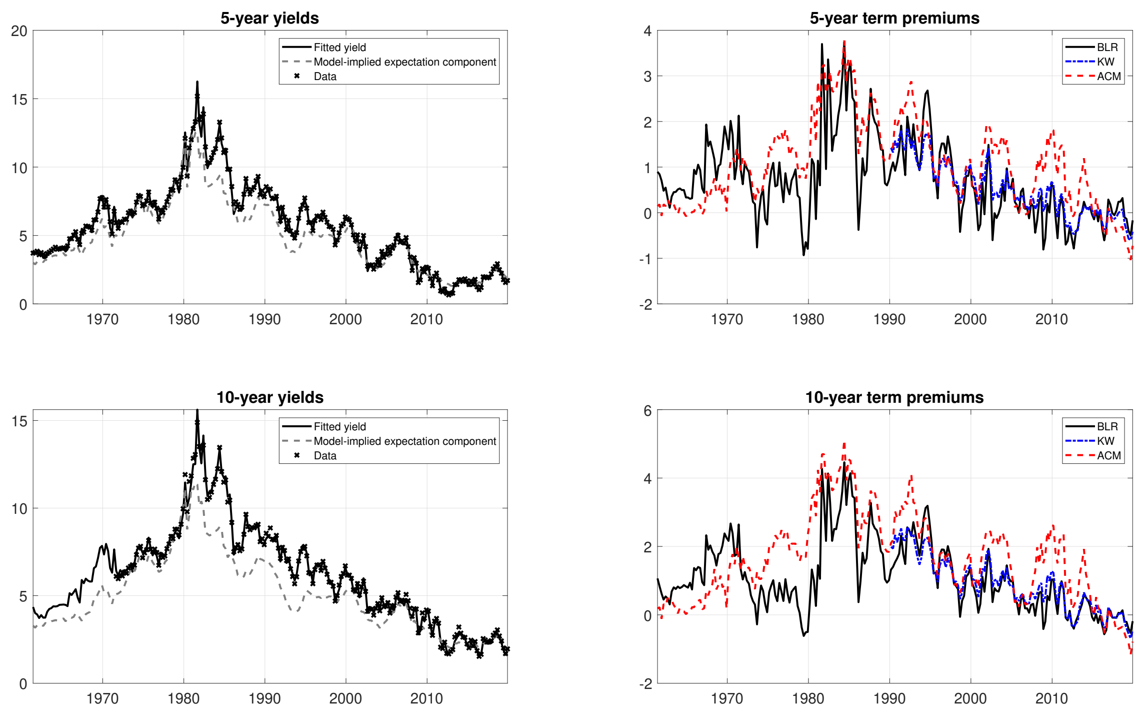
	Nominal term prem.			Real term prem.			Inflation risk prem.		
	2 yrs	5 yrs	10 yrs	2 yrs	5 yrs	10 yrs	2 yrs	5 yrs	10 yrs
A. Unconditional means of term premiums									
baseline	1.15	1.82	2.03	0.32	0.72	1.12	0.82	1.10	0.90
constant RA	1.15	1.83	2.06	0.33	0.75	1.17	0.82	1.09	0.90
constant $corr(\pi, c)$	1.17	1.91	2.21	0.32	0.72	1.12	0.84	1.18	1.09
zero $corr(\pi, c)$	-0.07	-0.54	-1.12	0.32	0.72	1.12	-0.39	-1.27	-2.24
B. Unconditional standard deviation of term premiums									
baseline	1.04	1.61	1.54	0.16	0.29	0.33	0.92	1.38	1.25
constant RA	0.58	0.81	0.67	0.00	0.00	0.00	0.58	0.81	0.67
constant $corr(\pi, c)$	0.81	1.35	1.39	0.16	0.29	0.33	0.65	1.05	1.06
zero $corr(\pi, c)$	0.16	0.29	0.33	0.16	0.29	0.33	0.00	0.00	0.00

Notes: This table reports the unconditional means and standard deviations of premiums. In addition to the baseline version, it also shows how the moments are affected when risk aversion is constant at μ_γ (that is, $\sigma_w = \sigma_m = 0$) or when the conditional inflation-consumption correlation is constant at μ_κ (that is, $\sigma_k = 0$) or equals zero (that is, $\mu_\kappa = \sigma_k = 0$). The computation of the unconditional moments relies on the analytical formulas given in Online Appendix II.

our term premiums show similar volatility and persistence but at lower average magnitudes. In addition, our estimated term premium series does not show a strong trend component. This finding is in line with the results of Bauer and Rudebusch (2020) despite the fact that they explicitly model a trend in long-run rate expectations (time-varying i^*) while we have a stationary (yet persistent) model at work.

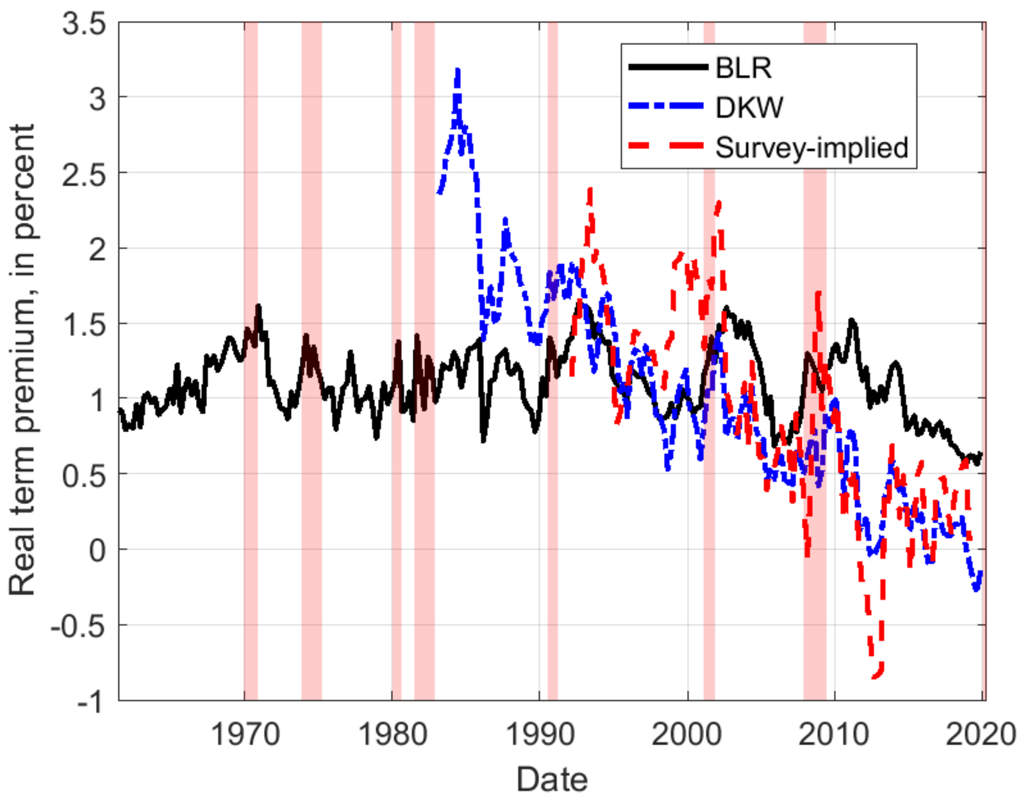
Figure 5 shows the estimated 10-year real term premium which has a broadly comparable volatility as those by D'Amico et al. (2018), yet with a less pronounced negative trend. Finally, inflation risk premiums displayed in Figure 6 are significantly more volatile than those by D'Amico et al. (2018), but both the level and the quarterly changes are highly co-moving with their estimate with a correlation of 0.9 over the common sample since 1983. The match in both figures with a survey-implied proxy is unsurprisingly closer as our model includes surveys in the estimation.

Figure 4: Model-implied nominal yields and associated term premiums



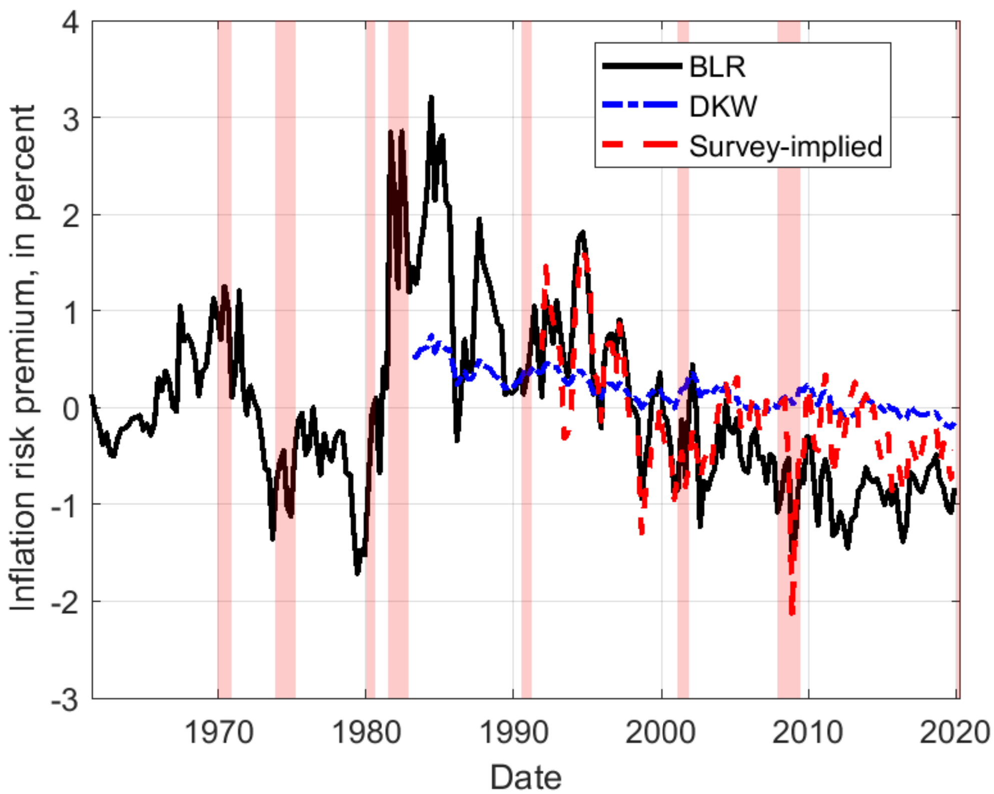
Notes: The figure displays the 5-year and 10-year model implied nominal term premiums (see Subsection 2.5 and in particular eq. 23). The right-hand-side plots compare our estimates to those obtained by Kim and Wright (2005) and Adrian et al. (2013).

Figure 5: 10-year real term premiums



Notes: This figure displays the 10-year real term premium defined as $TP_{t,h}$ in (23) alongside the estimate of D'Amico et al. (2018) and a survey-implied proxy, calculated from our data set in Table 1 as REALR10 - (BILL10 - CPI10). The red-shaded areas highlight NBER recessions. Sample period: 1961Q2-2019Q4.

Figure 6: 10-year inflation risk premiums



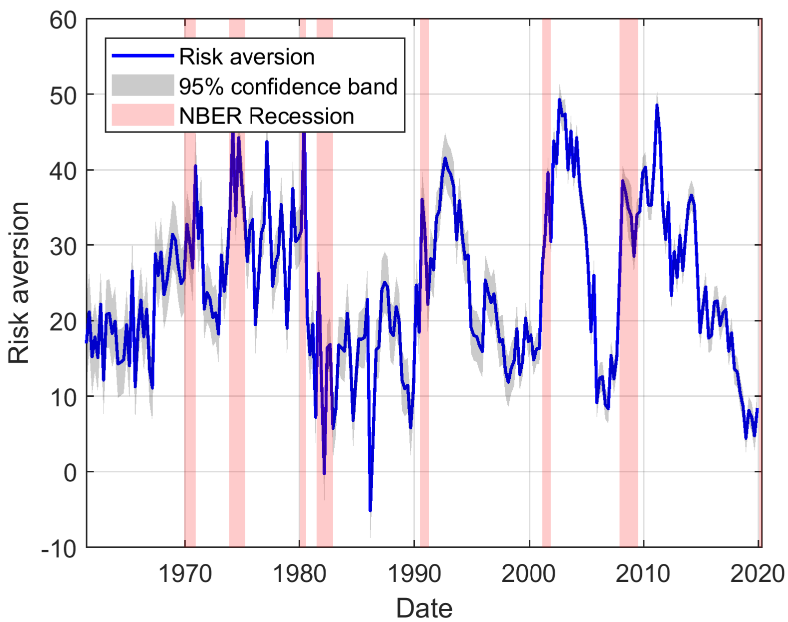
Notes: This figure displays the 10-year inflation risk premium, given by $TP^{\$}_{t,h} - TP_{t,h}$, where the nominal and real term premiums $TP^{\$}_{t,h}$ and $TP_{t,h}$ are defined in (23), alongside the estimate of D'Amico et al. (2018) and a survey-implied proxy, calculated from our data set in Table 1 as YIELD10 - REALR10 - CPI10. The inflation risk premium can be understood as the risk premium component of the inflation compensation $i_{t,h} - r_{t,h}$ (also called break-even inflation rate). The red-shaded areas highlight NBER recessions. Sample period: 1961Q2-2019Q4.

4.3 Risk aversion and term premium dynamics

The model-implied risk aversion exhibits clear time variation, as shown in Figure 7. When interpreting the dynamics and gauging its plausibility, it has to be noted that time series of risk aversion proxies in the literature often rely on stock market information, while our measure of risk aversion is inferred from the interaction of bond yields and macro information. Nevertheless, the overall dynamics are fairly similar to related measures in the literature, see Figure 8. Our measure co-moves negatively (absolute correlation of 0.5) with the measure of *risk perception* proposed by Pflueger et al. (2019) and similarly with the measure of *risk appetite* proposed by Bauer et al. (2023). Overall, we judge our filtered risk aversion as exhibiting relatively plausible dynamics. The average level of estimated risk aversion is 24 and is therefore in the same ballpark of other studies: for instance, Bansal and Shaliastovich (2013) obtain a level of around 20.

Without using stock market data, our risk aversion captures the buoyant US stock market in the sixties (low risk aversion, low excess returns), high levels of risk aversion in the turbulent 1970s, a low risk aversion and low equity premiums in the 1980s and 1990s, followed by the reversal coinciding with the burst of the tech-stock boom in the early 2000s and in the aftermath of the global financial crisis as of 2008. The negative trend in our risk aversion measure as of 2010 could reflect the large-scale asset purchases of the Federal Reserve which contributed to a compression of term premiums (see, e.g., Li and Wei, 2013). Lower premiums due to quantitative easing is not accounted for in our model and thus shows up in lower risk aversion. The biggest discrepancy with other measures of risk aversion is the lack of the upward spike at the height of the global financial crisis in 2008. In fact, our model would produce a big drop—rather than a spike—in risk aversion due to the collapse of quarter-on-quarter inflation and simultaneous surges in liquidity premiums for TIPS bonds if we had not set higher standard deviations for the measurement errors of inflation and real yields in the second half of 2008 (see Section 3). D'Amico et al. (2018) estimate that TIPS yields exceeded risk-free real yields by up to 300 basis points during the 2007–2008 financial crisis, which our model cannot control for. Likewise, Fleckenstein et al. (2014) document a significantly higher mispricing of TIPS in that period. In addition, the financial crisis was arguably both a reassessment of the amount of risk to be absorbed by the market and of preferences to be exposed to risk. Our measure only captures the second aspect.

Figure 7: Risk aversion coefficient



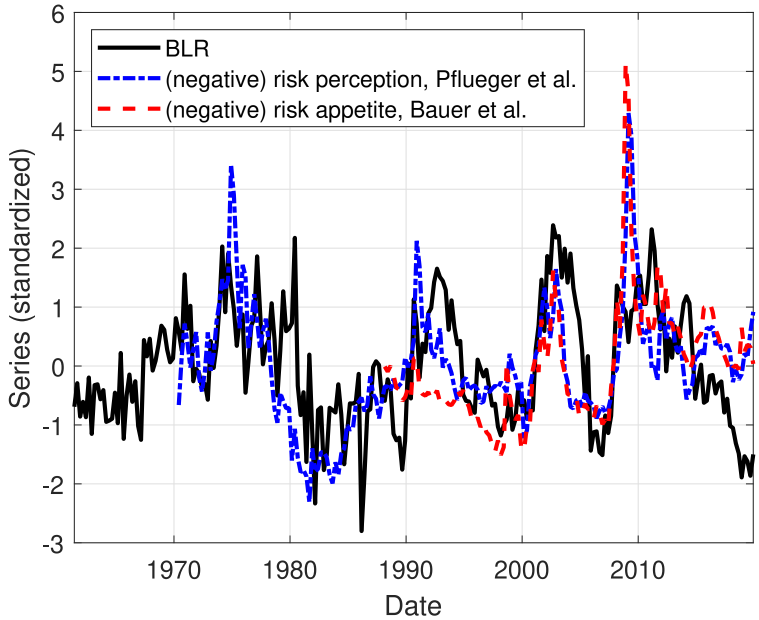
Notes: This figure shows the estimated risk aversion coefficient (γ_t in eq. 9). The red-shaded areas highlight NBER recessions. Sample period: 1961Q2-2019Q4. Confidence bands reflect filtering uncertainty.

Risk aversion is a key driver of bond pricing in our model. Figure 9 shows the close relationship between risk aversion and real term premiums. The difference in dynamics of real term premiums across maturities results from the two-factor structure of risk aversion in eq. 10. The faster-moving component, w_t, fades out more quickly and is thus less relevant than the slow-moving component, m_t, for the pricing of longer-term bonds.

The relevance of risk aversion for the dynamics of our model is also visible in impulse response functions. Figure 10 illustrates that a change in risk aversion (a materialisation of the shock $\varepsilon_{w,t}$) is a key driver of real term and inflation risk premiums (and so of nominal term premiums).[12] Our linear-quadratic specification of the nominal side of the economy makes the dynamics of nominal bond prices in response to the risk aversion shock dependant on the state of the economy. The three columns in Figure 10 differ in the starting value of the state driving the inflation-consumption

[12] The results here hold equally for the slow-moving component, $\varepsilon_{m,t}$, in risk aversion—just with higher persistence.

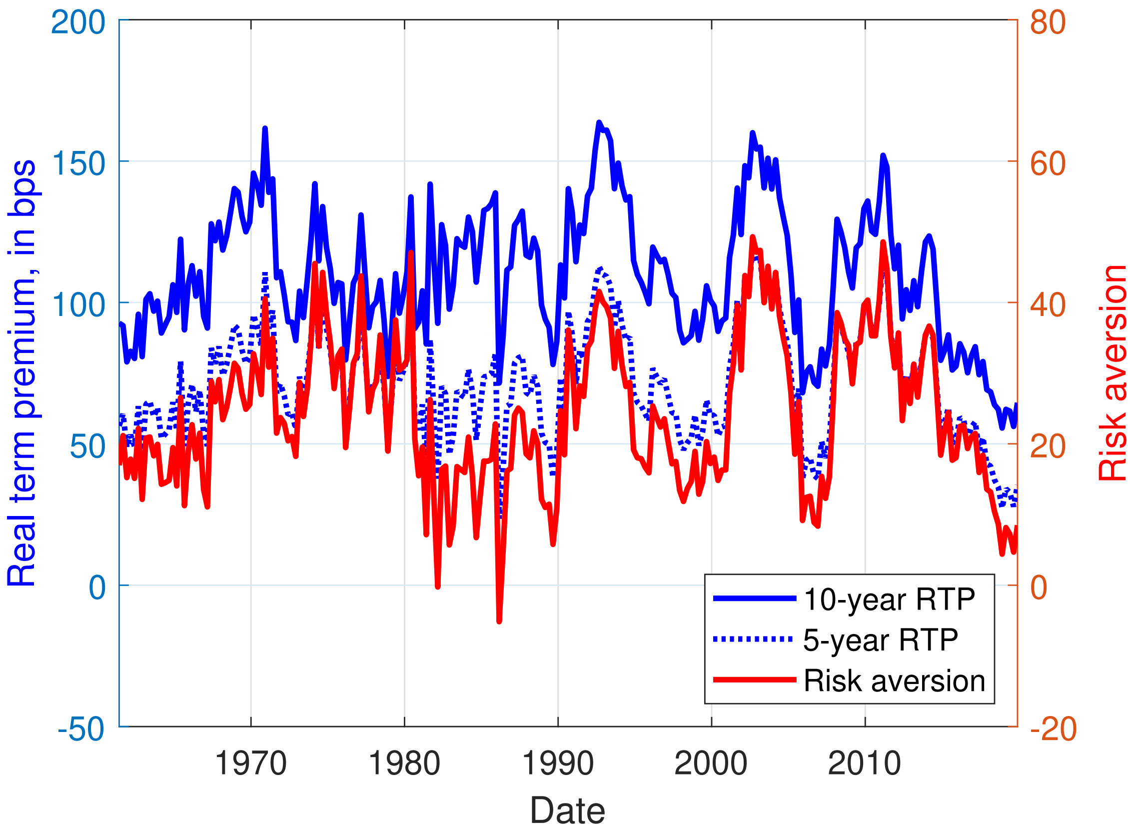
Figure 8: Risk aversion coefficient - comparison with Excess CAPE and Pflueger et al. risk appetite



Notes: This figure shows the estimated risk aversion coefficient (γ_t in eq. 9) together with (minus) the risk appetite index of Bauer et al. (2023) and (minus) the risk perception index of Pflueger et al. (2019). All series are standardised to make the original units comparable.

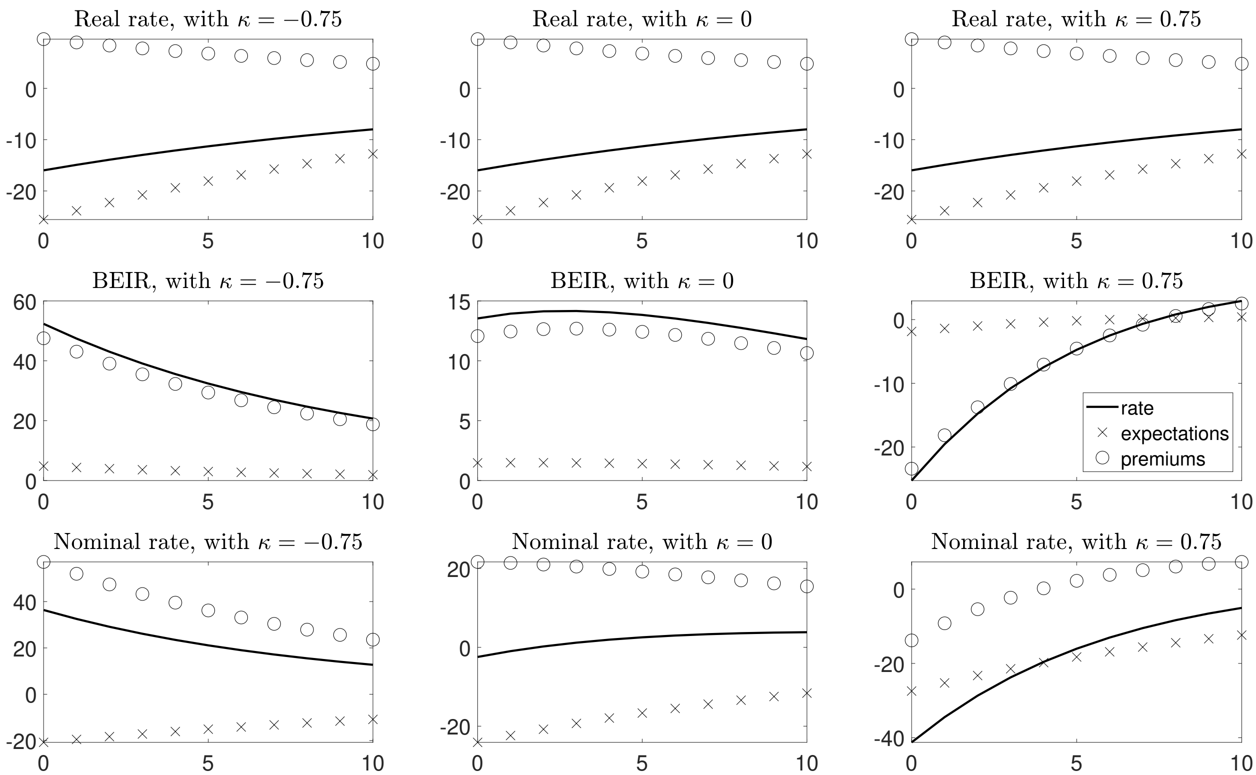
correlation, κ_t, on impact of the shock. While the effect on real rates is indifferent to the state (first row), the effect on break-even inflation rates (defined as nominal minus real rates) hinges critically on the initial value of κ_t (second row). A strongly negative value (here equal to -0.75) implies a supply-driven environment, as inflation and consumption growth correlate negatively, in which inflation risk premiums are already positive and get even higher in response to higher risk aversion (second row, left panel). By contrast, in a demand-driven environment with a strongly positive κ_t (here 0.75), inflation risk premiums are negative and get even lower (second row, right panel). In a balanced supply-demand environment, risk aversion has only little effects on inflation risk premiums even though they are not entirely zero due to non-linear effects (second row, middle panel). The effect on nominal rates eventually reflects the joint effect on real and break-even inflation rates (third row). Overall, risk aversion works as amplifier of inflation risk premiums in our model: it makes negative inflation risk premiums more negative and positive inflation risk premiums more positive—an intuitive feature that is not replicable with a purely linear model (see also Subsection 4.2).

Figure 9: Real term premiums versus risk aversion



Notes: This figure compares the 5-year and 10-year real term premium to the filtered risk aversion (γ_t). The real term premiums $TP_{t,h}$ are defined in (23). Sample period: 1961Q2-2019Q4.

Figure 10: State-dependant impulse responses of rates to a risk aversion shock

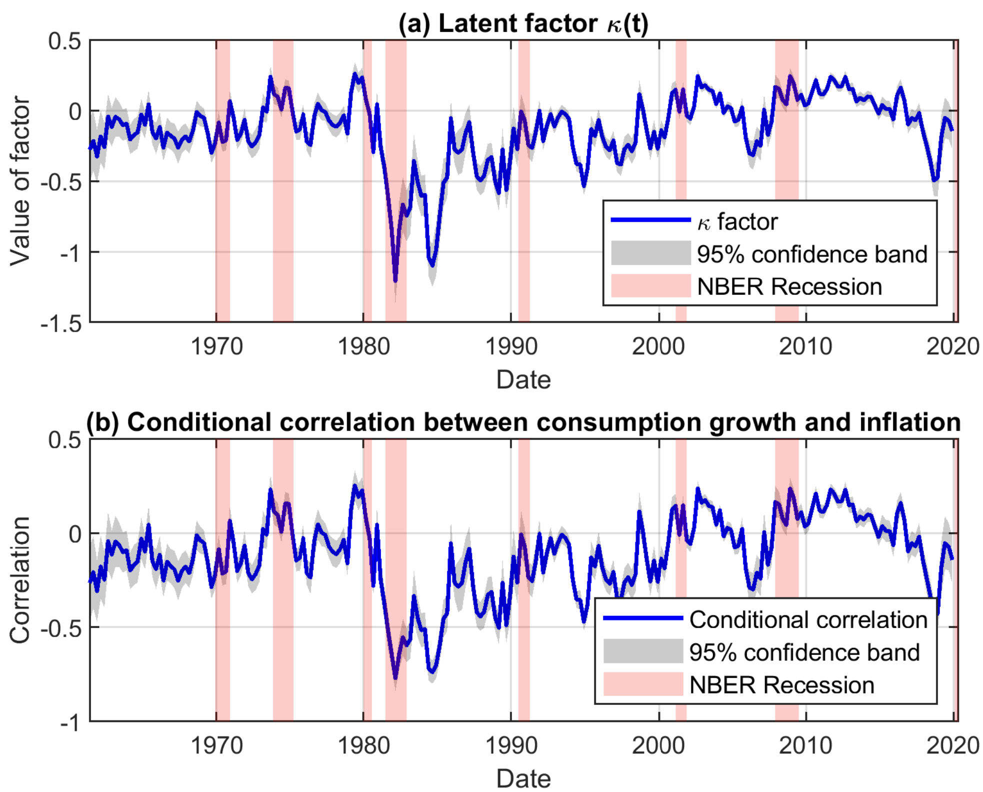


Notes: This figure shows the dynamic effects of one-unit increases (equal to one standard deviation) in the fast-moving risk aversion shock, ε_w, on 10-year real, break-even inflation and nominal rates. Effects are expressed in basis points (y-axis). The x-axis shows the number of quarters after the shock. The black solid lines correspond to the total effect of the shock on each rate. Crosses and circles show the two components of these effects: on the expectations component (crosses) and on the risk-premium component (circles). The three columns differ in the starting value of the state κ_t on impact of the shock. All other states are set to their unconditional mean. More precisely, the plots show the differences between two types of conditional expectations h quarters ahead: considering a given rate of interest x_t (real, nominal, or BEIR), the first conditional expectation is $\mathbb{E}(x_{t+h}|\kappa_t = \bar{\kappa}, \varepsilon_{w,t} = 1)$ and the second is $\mathbb{E}(x_{t+h}|\kappa_t = \bar{\kappa})$ with $\bar{\kappa} \in [-0.75, 0, +0.75]$ from left to right columns.

4.4 Inflation-consumption correlation and inflation risk premium dynamics

Our model incorporates a mechanism that allows for a time-varying prominence of supply vs demand shocks driving consumption growth and inflation. This is implemented via the (latent) variable κ_t that is proportional to the conditional covariance of inflation and consumption growth, see again Equation (7) above. Figure 11 displays the filtered κ_t series in panel (a) and the one-step-ahead conditional correlation of consumption growth and inflation in panel (b).[13]

Figure 11: Conditional correlation between consumption growth and inflation



Notes: This figure displays the estimate of factor κ_t, which drives the correlation between inflation (π_t) and consumption growth (Δc_t). See Subsection 2.1 for modeling details. The bottom plot shows the model-implied one-step-ahead conditional correlation between π and Δc_t. The red-shaded areas highlight NBER recessions. Sample period: 1961Q2-2019Q4. Confidence bands reflect filtering uncertainty (obtained by the delta method for the lower plot).

Over our sample, the evolution of this correlation can be grouped roughly into three phases: first, a near-zero correlation in the beginning; second, a steep decline towards a negative correlation in the 1980s; third; starting in the 1990s, a normalisation of the correlation to zero and eventually

[13] The correlation between the two panels is very high, yet not perfect as the second panel is the correlation (rather than the covariance). See the discussion around equation 7.

to positive levels, indicating a dominance of demand shocks driving the economy. The start of the second phase coincides with the second oil crisis in 1979, in which surging oil prices drove up inflation and depressed growth in the form of a classical negative cost-push supply shock. The most negative levels of the inflation-consumption correlation in the second phase can be attributed to the Volcker disinflation era. Even though this era is typically not considered a supply shock, the policy of the Federal Reserve under Volcker contributed to a strong and long-lasting decrease of (structural) inflation without entirely stalling the economy as feared back in the time (Goodfriend and King, 2005).

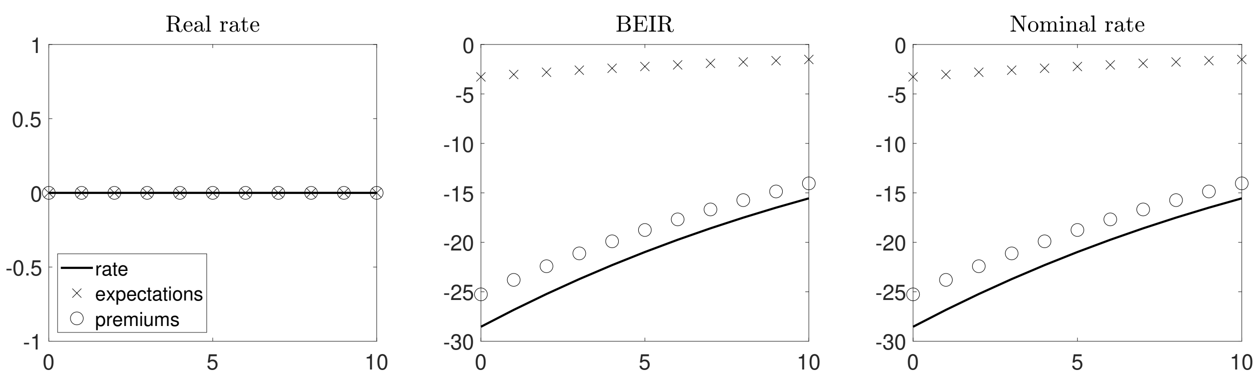
Let us stress that the ability of our approach to infer the evolution of κ_t, and hence of the conditional covariance between consumption and inflation, strongly hinges on observed variation in real and nominal bond yields, that are supposed to capture the investors' perception of forward-looking co-movements in macroeconomic variables. Our filtering procedure is indeed such that κ_t estimates are flat when macroeconomic variables alone—consumption growth, output gap and inflation measurements—are used as observations. This is because, as is the case for other modified Kalman filtering approaches proposed in the literature (e.g., de Jong, 2000; Duan and Simonato, 1999; Monfort et al., 2017), our QKF-based approach can detect movements in the latent factors only if the latter affect the conditional means of observed variables.[14] By contrast to macroeconomic variables, the conditional means of yields depend on κ_t, which makes it possible to infer its variations by means of our filtering approach.

The model contains an intimate link between the time-varying relevance of supply vs demand shocks on the one hand, and inflation risk premiums on the other hand, confirming that inflation risk premiums are positive amid the prevalence of supply shocks and negative when demand shocks dominate the economy. One way to see this nexus operating in our model economy is via the

[14] To get the intuition behind that result, consider a simple state-space model whose measurement equations are $y_t = Bk_t + \Sigma(k_t)\varepsilon_t$, where y_t is the vector of observed variables, and where k_t follows an AR(1) process. Assume we use a modified Kalman filter (as is the case here, see Section 3 and Appendix B) to estimate k_t. The Kalman gain then is a product of matrices, one of them being matrix B. Hence, if $B = 0$, i.e., if k_t affects observed variables only through second order moments, the gain is 0, and the Kalman estimate of k_t converges to its unconditional mean. This relates to the fact that the (modified) Kalman filter we use is not optimal in this heteroskedastic context. Other nonlinear filters (e.g. a particle filter) may be used to exploit the additional information contained in the macroeconomic variables' innovations; this is left for further research.

impulse responses in Figure 12: a positive realisation of the shock $\varepsilon_{k,t}$ (leading to an increase in κ and thus a higher inflation-consumption correlation, i.e., letting the economy look more demand-dominated) leaves the real yield curve unaffected, but decreases inflation risk premiums and hence nominal term premiums and bond yields. This comparison bears similarity to Breach et al. (2020) who also focus on the relation between $\mathbb{C}ov_t(\pi_{t+h}, \Delta c_{t+h})$ and inflation risk premiums, likewise stressing that *"in a world where supply shocks dominate, this covariance is strongly negative, investors fear inflation, and the risk premium for bearing inflation risk is positive"*, and the other way round for demand shocks. They co-plot their estimated inflation risk premium estimates with correlation measures of expected growth and inflation, using the correlation of stock prices and (synthetic) TIPS-based break-even inflation rates as proxies. They confirm the inverse relation between the two. Unlike in their paper that uses off-model inflation-consumption correlations, our model simultaneously generates inflation risk premiums and time-varying macro correlations within one framework.

Figure 12: Impulse responses of rates to a $corr(\pi, c)$ shock

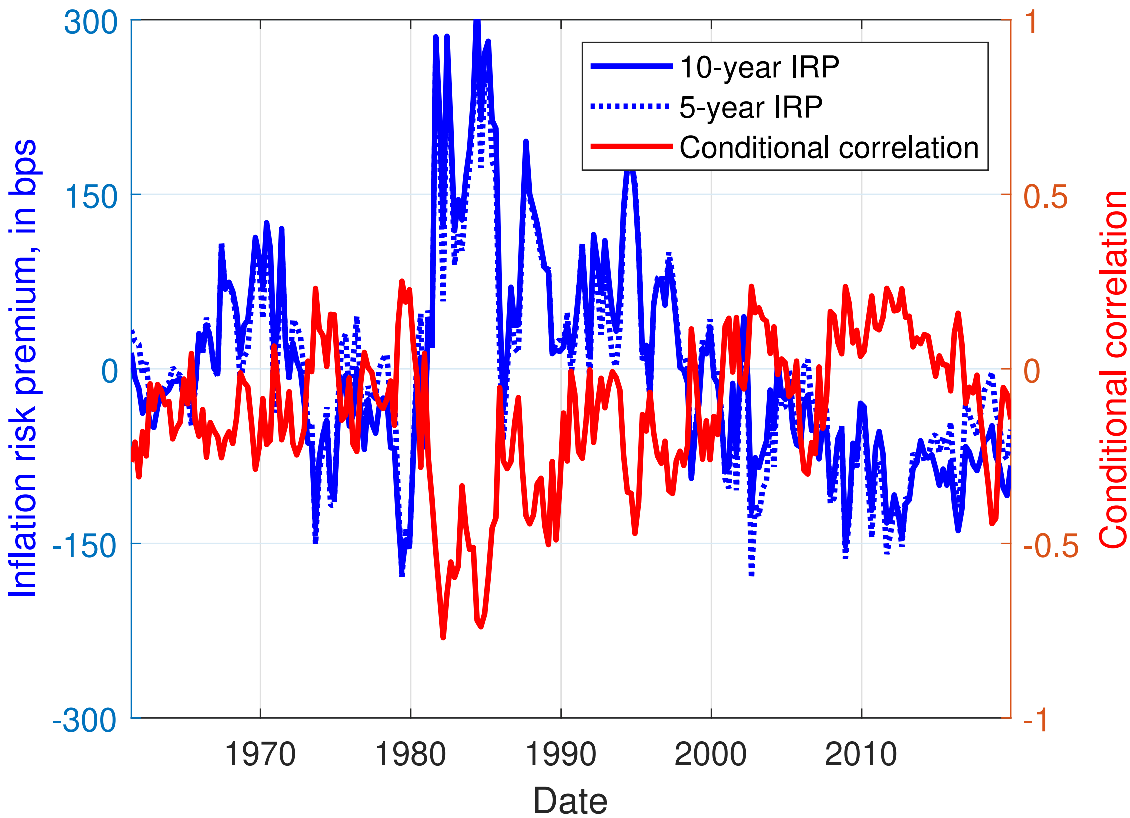


Notes: This figure shows the dynamic effects of one-unit increases (equal to one standard deviation) in the shock driving the inflation-consumption correlation, ε_k, on 10-year real, break-even inflation and nominal rates. Effects are expressed in basis points (y-axis). The x-axis shows the number of quarters after the shock. The black solid lines correspond to the total effect of the shock on each rate. Crosses and circles show the two components of these effects: on the expectations component (crosses) and on the risk-premium component (circles). On impact of the shock, all states are set to their unconditional mean. More precisely, considering a given rate of interest x_t (real, nominal, or BEIR), the plots show the conditional expectation $\mathbb{E}(x_{t+h}|\varepsilon_{k,t} = 1)$ for h quarters ahead.

Another, more direct, evidence is given by the co-plot of the time-varying inflation-consumption correlation (same as in Figure 11) and model-implied inflation risk premiums, see Figure 13. There is a clear inverse co-movement between the two in line with the economic intuition. At the same time, inflation risk premiums are not the exact mirror image of the time-varying inflation-consumption correlation as they are also driven by time variation in risk aversion and by the interaction between the correlation and risk aversion (see again Subsection 4.2).

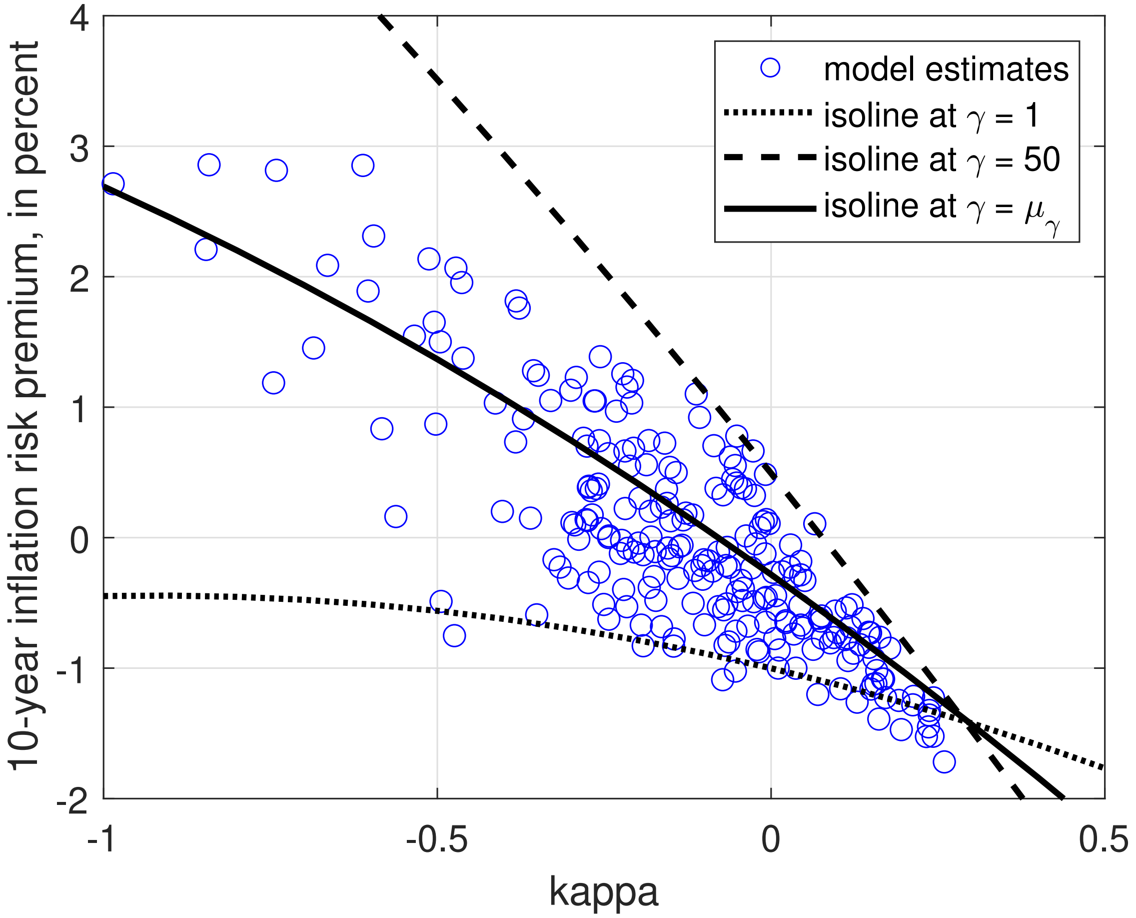
The importance of risk aversion for the variation in inflation risk premiums and as amplifier of the time-varying inflation-consumption-correlation driver is also illustrated in Figure 14. If the inflation-consumption correlation was the only source of variation in inflation risk premiums, the points in the scatter plot between the driver of the inflation-consumption correlation, κ, and the associated inflation risk premiums would all line up on a smooth line invariant to the level of risk aversion. Note that such line need not be linear due to the linear-quadratic nature of the model. Yet, the model estimates are scattered between two "isolines" of risk aversion, i.e. model-implied mappings between κ and the 10-year inflation risk premium for a fixed level of risk aversion. One such isoline belongs to a very low level of risk aversion of 1, the other to a high value of 50. For example, near a level of $\kappa = -0.5$, the isolines span a range of 400 basis points for the corresponding inflation risk premiums and the actually estimated inflation risk premiums fluctuate within a range of over 250 basis points.

Figure 13: Inflation risk premiums versus the conditional inflation-consumption correlation



Notes: This figure compares the 5-year and 10-year inflation risk premiums to the conditional correlation between inflation (π_t) and consumption growth (Δc_t). The inflation risk premium of maturity h is given by $TP^{\$}_{t,h} - TP_{t,h}$, where the nominal and real term premiums $TP^{\$}_{t,h}$ and $TP_{t,h}$ are defined in (23); the inflation risk premium can be understood as the risk premium component of the inflation compensation $i_{t,h} - r_{t,h}$ (also called break-even inflation rate). Sample period: 1961Q2-2019Q4.

Figure 14: Amplification of inflation risk premiums via risk aversion



Notes: This figure shows the scatter plot between the estimated 10-year inflation risk premiums and the filtered κ_t which drives the conditional inflation-consumption correlation. The isolines show the model-implied inflation risk premium as a function of κ for different values of risk aversion, γ, by setting the states determining κ and γ accordingly and keeping all other states at their unconditional means. As risk aversion can be obtained by any arbitrary combination of its fast- and slow-moving components, w_t and m_t in eq. 10, the relative split is based on each component's contribution to the unconditional variance of risk aversion. Blue circles outside the isolines result from either more extreme values of risk aversion or a different composition of the risk aversion components.

5 Conclusion

The paper contributes to the macro-finance literature with a new equilibrium term structure model that features Epstein-Zin utility, time-varying risk aversion, a trend-cycle decomposition of consumption, hysteresis effects, and a time-varying consumption-inflation correlation. The latter renders the model linear-quadratic, as opposed to completely affine, but we still obtain (linear-quadratic) closed-form solutions for nominal bond yields, real yields, inflation compensation and associated premiums. The model is estimated on US data, including macro variables (inflation and consumption), nominal and real bond yields, and survey expectations on future macro and financial variables.

The model reproduces several empirical patterns that are central to macro-finance analysis. For instance, it sheds light at the often alleged nexus between inflation risk premiums and the prominence of supply vs demand shocks: *"At times when inflation is countercyclical as will be the case if the economy is affected by supply shocks [...] nominal bond returns are procyclical and investors demand a positive risk premium to hold them"*, Campbell et al. (2017), and vice versa for demand shocks. Via the assumed time-varying consumption-inflation correlation, the model implies that supply-driven episodes coincide with positive inflation risk and nominal term premiums, while demand-driven episodes coincides with negative/lower premiums. Time-varying risk aversion is identified as an additional factor that can impact these premiums.

As another appealing property, our specification of consumption dynamics—trend cycle decomposition with hysteresis effects—allows the model to generate a real curve that is upward-sloping on average, i.e. it generates positive unconditional expectations of real term premiums. This is feature in the data that several structural macro-finance yield curve models struggle to generate.

The model lends itself to being used as a laboratory, e.g. in central bank analysis, for studying the interplay of consumption and inflation dynamics with bond and inflation risk premiums. However, linking bond and inflation risk premiums to specific preferences and associated macro dynamics in an equilibrium model like ours comes with a trade-off. On the one hand, using our model links estimation directly with an economic narrative: it provides a coherent framework for both estimating risk premiums and linking their dynamics to risk aversion and macro drivers. On the other hand, the structural restrictions that come with such an equilibrium approach and the restricted number of factors allow for less flexibility in fitting nominal and real yield curves compared to, say, flexible (arbitrage-free) approaches working purely latent factors.

In future work, it may be worthwhile to assess some of our model properties in more depth and refine them. For instance, it is intriguing that the dynamics of our estimated risk aversion parameter are close to that of other approaches in the literature, despite the fact that our approach does not utilize stock market information while most others do. The reason for this pattern could be explored more carefully. As another example and as mentioned in the text, we may use a simulation-based or particle filter in order to broaden the inference on the time-varying consumption-inflation correlation to cover macro data. This extensions would, however, substantially complicate the model estimation. Finally, it would be interesting to apply the model to the euro area that, despite its relatively short history, has arguably also displayed the change in sign of inflation risk premiums and the associated time variation in supply vs demand shocks driving the economy. We leave this extensions for future research.

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A Parameterization of the Gaussian linear-quadratic model

The matrices defining the Gaussian linear-quadratic model outlined in Subsection 2.3 are:

$$\Phi=\left[\begin{array}{cccccc}\rho_g & \rho_{gz} & 0 & 0 & 0 & 0\\ 0 & \rho_z & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \rho_w & 0 & 0\\ 0 & 0 & 0 & 0 & \rho_m & 0\\ 0 & 0 & 0 & 0 & 0 & \rho_k\end{array}\right],\ \Sigma=\left[\begin{array}{ccccccc}\sigma_g & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}}\sigma_z & \frac{1}{\sqrt{2}}\sigma_z & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \sigma_w & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \sigma_m & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \sigma_k & 0\end{array}\right],$$

$$\Phi_Z=\left[\begin{array}{cc}\rho_\pi^* & 0\\ 0 & \rho_{\tilde\pi}\end{array}\right],$$

and

$$vec\left(\Sigma_Z(X_t)\right) = \underbrace{\left[\begin{array}{c}0\\0\\0\\\frac{1+\mu_\kappa}{2}\sigma_{\pi,z}\\0\\-\frac{1-\mu_\kappa}{2}\sigma_{\pi,z}\\0\\0\\0\\0\\0\\0\\\sigma_\pi^*\\0\end{array}\right]}_{=\Gamma_0}+\underbrace{\left[\begin{array}{cccccc}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&\frac{1}{2}\sigma_{\pi,z}\\0&0&0&0&0&0\\0&0&0&0&0&\frac{1}{2}\sigma_{\pi,z}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{array}\right]}_{=\Gamma_1}X_t.$$

B State-space representation

Let us denote by U_t the vector of observations on date t (with some components that may be missing values). This vector can gather: macroeconomic variables, nominal and real yields, survey-based data. According to the model, such variables are affine functions of Y_t. That is, the measurement equations admit the following representation:

$$U_t = A_U + B_U Y_t + \eta_t,$$

where $\eta_t \sim i.i.d.\ \mathcal{N}(0,\Omega_\eta)$ is a vector of Gaussian measurement errors.

The transition equation is given by the vector auto-regressive representation of Y_t (see Subsection 2.3). The existence of this representation stems from the affine property of the extended state vector, see (II.3) in Online Appendix II.

The model can be estimated by means of the Kalman filter techniques, maximizing an approximation to the likelihood function. The Kalman filter is not optimal because $\mathbb{V}ar_t(Y_{t+1})$ is not constant in our context, and because Y_t's innovations are not Gaussian. As shown in Corollary 1, this conditional variance depends linearly on Y_t. This is easily accommodated in a modified Kalman filter, as, e.g., in de Jong (2000), Duan and Simonato (1999), or Monfort et al. (2017). Contrary to the case where square-root processes are involved, we do not have to deal with the sign of the latent factors (the components of X_t and Z_t), as all these factors are real-valued. However, on each iteration of the Kalman filter, we need to deal with the fact that some components of Y_t are deterministic functions of the other ones: indeed, the last $n_X(n_X+1)/2$ entries of Y_t correspond to $vech(X_tX_t')$, where X_t correspond to the first n_X entries of Y_t. In practice, following Monfort et al. (2015), at the end of the updating step, we replace the last $n_X(n_X+1)/2$ entries of the updated vector ($Y_{t|t}$) with $vech(X_{t|t}X_{t|t}')$, where vector $X_{t|t}$ gathers the first n_X components of $Y_{t|t}$.

C General econometric setting

C.1 The real side of the economy

Assumption A.1. *The preferences of the representative agent are of the Epstein and Zin (1989) type, with a unit elasticity of intertemporal substitution (EIS). Specifically, the time-t log utility of a consumption stream (C_t) is recursively defined by*

$$u_t \quad = \quad \log U_t = (1-\delta)c_t + \frac{\delta}{1-\gamma_t}\log\left(\mathbb{E}_t \exp\left[(1-\gamma_t)u_{t+1}\right]\right), \tag{24}$$

where c_t denotes the logarithm of the agent's consumption level C_t, δ the pure time discount factor and γ_t is the risk aversion parameter.

Assumption A.2. *The log growth rate of consumption (Δc_t) as well as the coefficient of risk aversion (γ_t) are affine in an n_X-dimensional vector of state variables X_t:*

$$\Delta c_t = \mu_{c,0} + \mu_{c,1}' X_t \tag{25}$$

$$\gamma_t = \mu_{\gamma,0} + \mu_{\gamma,1}' X_t. \tag{26}$$

X_t follows a Gaussian VAR(1) process, that is:

$$X_t = \Phi X_{t-1} + \Sigma \varepsilon_t, \tag{27}$$

where $\varepsilon_t \sim i.i.d. \mathcal{N}(0, I_{n_\varepsilon})$. (Matrix Σ is of dimension $n_X \times n_\varepsilon$.) This implies that

$$\mathbb{E}_t(\exp(w'X_{t+1})) = \exp\left(\psi_0(w) + \psi_1(w)'X_t\right)$$

with:

$$\psi_0(w) = \frac{1}{2}w'\Sigma\Sigma'w \quad \textit{and} \quad \psi_1(w) = \Phi'w.$$

The following proposition exhibits the s.d.f. that prevails under the previous assumptions.

Proposition 1. *Under Assumptions A.1 and A.2, the s.d.f. is given by*

$$\mathscr{M}_{t,t+1} \quad = \quad \exp\left[-(\eta_0 + {\eta_1}'X_t) + \lambda_t'X_{t+1} - \lambda_t'\Phi X_t - \frac{1}{2}\lambda_t'\Sigma\Sigma'\lambda_t\right],$$

where

$$\left\{\begin{array}{lcl} \eta_0 & = & -\log(\delta) + \mu_{c,0} + \frac{1}{2}\mu_{c,1}'\Sigma\Sigma'\mu_{c,1} + \lambda_0'\Sigma\Sigma'\mu_{c,1} \\ \eta_1 & = & (\lambda_1\Sigma\Sigma' + \Phi')\,\mu_{c,1}, \end{array}\right. \tag{28}$$

and $\lambda_t = \lambda_0 + \lambda_1'X_t$*, with:*

$$\left\{\begin{array}{lcl} \lambda_0 & = & \left[(1-\mu_{\gamma,0})\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' - \mu_{\gamma,0}I_{n_X}\right]\mu_{c,1} \\ \lambda_1 & = & -(\mu_{\gamma,1}\mu_{c,1}')(\delta\Phi(I_{n_X} - \delta\Phi)^{-1} + I_{n_X}). \end{array}\right. \tag{29}$$

The short-term real rate r_t *is affine in* X_t*:*

$$r_t = -\log[\mathbb{E}_t(\mathscr{M}_{t,t+1})] = \eta_0 + \eta_1'X_t. \tag{30}$$

Proof. See Appendix I.1. □

C.2 Inflation and extended state vector

Assumption A.3. *The (log) inflation rate is given by:*

$$\pi_t = \mu_{\pi,0} + \mu_{\pi,Z}'Z_t + \mu_{\pi,X}'X_t + \mu_{\pi,XX}'vech(X_tX_t'), \tag{31}$$

where

$$\begin{bmatrix} X_t \\ Z_t \end{bmatrix} = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi_Z \end{bmatrix}\begin{bmatrix} X_{t-1} \\ Z_{t-1} \end{bmatrix} + \begin{bmatrix} \Sigma \\ \Sigma_Z(X_{t-1}) \end{bmatrix}\varepsilon_t, \tag{32}$$

with $\varepsilon_t \sim i.i.d.\mathscr{N}(0, I_{n_\varepsilon})$ *(consistently with eq. 27), and with*

$$vec\,(\Sigma_Z(X_t)) = \Gamma_0 + \Gamma_1 X_t. \tag{33}$$

Since $\Sigma_Z(X_t)$ *is of dimension* $n_Z \times n_\varepsilon$*,* Γ_0 *and* Γ_1 *are of dimension* $(n_Z n_\varepsilon) \times 1$ *and* $(n_Z n_\varepsilon) \times n_X$*, respectively.*

Proposition 2. *Under Assumptions A.2 and A.3, process* Y_t *(with* $Y_t = [X_t', Z_t', vech(X_tX_t')']'$*)*

is affine. More precisely, we have:

$$\begin{aligned} \mathbb{E}_t(\exp(u'Y_{t+1})) &= \exp\left(\psi_{Y,0}(u) + \psi_{Y,1}(u)'Y_t\right) \qquad (34) \\ &= \exp\left[\psi_{Y,0}(u) + \psi_{Y,X}(u)'X_t + \psi_{Y,Z}(u)'Z_t + \psi_{Y,XX}(u)'vech(X_tX_t')\right], \end{aligned}$$

where $u = [u_X', u_Z', u_{XX}']'$, *and where functions* $\psi_{Y,0}$, $\psi_{Y,X}$, $\psi_{Y,Z}$, *and* $\psi_{Y,XX}$ *are defined by:*

$$\begin{aligned} \psi_{Y,0}(u) &= -\frac{1}{2}\log|I_{n_\varepsilon} - 2V(u)| + \frac{1}{2}v_0(u)'(I_{n_\varepsilon} - 2V(u))^{-1}v_0(u) \\ \psi_{Y,X}(u) &= \Phi' u_X + v_1(u)'(I_{n_\varepsilon} - 2V(u)')^{-1}v_0(u) \\ \psi_{Y,Z}(u) &= \Phi_Z' u_Z \\ \psi_{Y,XX}(u) &= K_{n_X}\left[(\Phi'\otimes\Phi')(M_{n_X}u_{XX}) + \frac{1}{2}vec\left(v_1(u)'(I_{n_\varepsilon} - 2V(u)')^{-1}v_1(u)\right)\right], \end{aligned}$$

with

$$\begin{aligned} v_0(u) &= \Sigma' u_X + (I_{n_\varepsilon}\otimes u_Z')\Gamma_0 \\ v_1(u) &= (I_{n_\varepsilon}\otimes u_Z')\Gamma_1 + 2\Sigma' vec^{-1}(M_{n_X}u_{XX})\Phi \\ V(u) &= vec^{-1}\{u_{XX}'M_{n_X}'(\Sigma\otimes\Sigma)\}, \end{aligned}$$

where

- M_{n_X} *is the matrix of dimension* $n_X^2 \times (n(n+1)/2)$ *that is such that, for any symmetric matrix* Ω, *and any vector* u *of dimension* $n_X(n_X+1)/2$, *we have* $u'vech(\Omega) = u'M_{n_X}'vec(\Omega)$;
- K_{n_X} *is the matrix of dimension* $n(n+1)/2 \times n_X^2$ *that is such that, for any symmetric matrix* Ω *of dimension* $n_X \times n_X$, *we have* $vec(\Omega) = K_{n_X}'vech(\Omega)$.

(Note: The three components of u *have dimensions that match those of* X_t, Z_t, *and* $vech(X_tX_t')$, *respectively.)*

Proof. Online Appendix I.3. □

C.3 Risk-neutral dynamics

The risk-neutral measure is defined with respect from the physical measure, by means of the Radon-Nikodym derivative $\mathscr{M}_{t,t+1}/\mathbb{E}_t(\mathscr{M}_{t,t+1})$.

Proposition 3. *Under Assumptions A.1 and A.2, the risk-neutral dynamics of* X_t *is:*

$$X_t = \mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}}X_{t-1} + \Sigma\varepsilon_t^{\mathbb{Q}},$$

where $\varepsilon_t^{\mathbb{Q}} \sim i.i.d.\mathcal{N}(0, I_{n_\varepsilon})$ *under* $\mathbb{Q}$*, and where:*

$$\mu^{\mathbb{Q}} = \Sigma\Sigma'\lambda_0 \quad \text{and } \Phi^{\mathbb{Q}} = \Phi + \Sigma\Sigma'\lambda_1',$$

λ_0 *and* λ_1 *being given in* (29).

Proof. See Appendix I.2. □

Proposition 4. *Under Assumptions A.1, A.2, and A.3, the risk-neutral dynamics of* Z_t *are of the form:*

$$Z_t = \mu_Z^{\mathbb{Q}} + \Phi_Z Z_{t-1} + \Phi_{ZX}^{\mathbb{Q}} X_{t-1} + \Phi_{ZXX}^{\mathbb{Q}} vec(X_{t-1}X_{t-1}') + \Sigma_Z(X_{t-1})\varepsilon_t^{\mathbb{Q}}, \tag{35}$$

where $\varepsilon_t^{\mathbb{Q}} \sim i.i.d.\mathcal{N}(0, I_{n_\varepsilon})$ *under* $\mathbb{Q}$*, and with*

$$\begin{aligned}
\mu_Z^{\mathbb{Q}} &= ([\Sigma^{-1}\mu^{\mathbb{Q}}]' \otimes I_{nz})\Gamma_0 \\
\Phi_{ZX}^{\mathbb{Q}} &= ([\Sigma^{-1}\mu^{\mathbb{Q}}]' \otimes I_{nz})\Gamma_1 + \begin{bmatrix} \Gamma_0' J_1' \Sigma^{-1}(\Phi^{\mathbb{Q}} - \Phi) \\ \vdots \\ \Gamma_0' J_{nz}' \Sigma^{-1}(\Phi^{\mathbb{Q}} - \Phi) \end{bmatrix} \\
\Phi_{ZXX}^{\mathbb{Q}} &= \begin{bmatrix} vec(\Gamma_1' J_1' \Sigma^{-1}(\Phi^{\mathbb{Q}} - \Phi))' \\ \vdots \\ vec(\Gamma_1' J_{nz}' \Sigma^{-1}(\Phi^{\mathbb{Q}} - \Phi))' \end{bmatrix},
\end{aligned}$$

where $\mu^{\mathbb{Q}}$ *and* $\Phi^{\mathbb{Q}}$ *are given in Prop. 3, and where, for* $i \in \{1, \dots, n_Z\}$, $J_i = I_{n_\varepsilon} \otimes e_{i,n_Z}'$*, where* e_{i,n_Z} *is the* i^{th} *column of the identity matrix of dimension* $n_Z \times n_Z$*.*

If Σ *is not invertible, then one can replace* Σ^{-1} *with the Moore-Penrose inverse of* Σ*.*

Proof. Online Appendix I.4. □

Proposition 5. *Under Assumptions A.1, A.2, and A.3, process* Y_t *is affine under the risk-neutral measure. More precisely, we have:*

$$\begin{aligned}
& \mathbb{E}_t^{\mathbb{Q}}(\exp(u'Y_{t+1})) \\
= \;& \exp\left[\psi_{Y,0}^{\mathbb{Q}}(u) + \psi_{Y,X}^{\mathbb{Q}}(u)'X_t + \psi_{Y,Z}^{\mathbb{Q}}(u)'Z_t + \psi_{Y,XX}^{\mathbb{Q}}(u)'vech(X_tX_t')\right].
\end{aligned} \tag{36}$$

where $u = [u_X', u_Z', u_{XX}']'$ *(the three components of* u *having dimensions that match those of*

X_t, Z_t, and $vech(X_tX_t')$, respectively), with:

$$
\begin{aligned}
\psi^{\mathbb{Q}}_{Y,0}(u) &= u_X'\mu^{\mathbb{Q}} + u_{XX}'vec(\mu^{\mathbb{Q}}\mu^{\mathbb{Q}'}) + u_Z'\mu_Z^{\mathbb{Q}} \\
&\quad -\frac{1}{2}\log|I_{n_\varepsilon} - 2V(u)| + \frac{1}{2}v_0^*(u)'(I_{n_\varepsilon} - 2V(u))^{-1}v_0^*(u) \\
\psi^{\mathbb{Q}}_{Y,X}(u) &= \Phi^{\mathbb{Q}'}u_X + 2(\Phi^{\mathbb{Q}'}\otimes\mu^{\mathbb{Q}'})(M_{n_X}u_{XX}) + \Phi^{\mathbb{Q}}_{ZX}{}'u_Z + v_1^*(u)'(I_{n_\varepsilon} - 2V(u)')^{-1}v_0^*(u) \\
\psi^{\mathbb{Q}}_{Y,Z}(u) &= \Phi_Z'u_Z \\
\psi^{\mathbb{Q}}_{Y,XX}(u) &= K_{n_X}\left[\Phi^{\mathbb{Q}}_{ZXX}{}'u_Z + (\Phi^{\mathbb{Q}'}\otimes\Phi^{\mathbb{Q}'})(M_{n_X}u_{XX}) + \frac{1}{2}vec\left(v_1^*(u)'(I_{n_\varepsilon} - 2V(u)')^{-1}v_1^*(u)\right)\right],
\end{aligned}
$$

with

$$
\begin{aligned}
v_0^*(u) &= \Sigma'u_X + 2(\Sigma'\otimes\mu^{\mathbb{Q}'})(M_{n_X}u_{XX}) + (I_{n_\varepsilon}\otimes u_Z')\Gamma_0 \\
v_1^*(u) &= (I_{n_\varepsilon}\otimes u_Z')\Gamma_1 + 2\Sigma' vec^{-1}(M_{n_X}u_{XX})\Phi^{\mathbb{Q}} \\
V(u) &= vec^{-1}\{u_{XX}'M_{n_X}'(\Sigma\otimes\Sigma)\}.
\end{aligned}
$$

Proof. Online Appendix I.5 □

D Pricing

Proposition 6. *Under Assumptions A.1 and A.2, the date-t price of a real zero-coupon bond of maturity h is given by:*

$$P_{t,h} = \exp\left(a_h + b_h{}'X_t\right), \tag{37}$$

with $a_0 = 0$ and $b_0 = 0$ and, for $h > 0$:

$$
\begin{aligned}
a_h &= -\eta_0 + a_{h-1} + b_{h-1}'\mu^{\mathbb{Q}} + \frac{1}{2}b_{h-1}'\Sigma\Sigma'b_{h-1} \\
b_h &= -\eta_1 + \Phi^{\mathbb{Q}'}b_{h-1}.
\end{aligned}
$$

Denoting by $r_{t,h}$ the maturity-h zero-coupon real yield, we have:

$$r_{t,h} = -\frac{1}{h}\log(P_{t,h}) = -\frac{1}{h}(a_h + b_h'X_t). \tag{38}$$

Proof. Make the conjecture that, for all date t and maturity h, there exist a scalar a_h and a vector b_h that are such that $P_{t,h}$ is of the form $P_{t,h} = \exp(a_h + b_h'X_t)$. Let's then compute a_{h+1} and b_{h+1}:

$$
\begin{aligned}
P_{t,h+1} &= \mathbb{E}_t^{\mathbb{Q}}(\exp(-\eta_0 - \eta_1'X_t + a_h + b_h'X_{t+1})) \\
&= \exp\left(-\eta_0 - \eta_1{}'X_t + a_h + b_h'\mu^{\mathbb{Q}} + \frac{1}{2}b_h'\Sigma\Sigma'b_h + b_h'\Phi^{\mathbb{Q}}X_t\right), \qquad (39)
\end{aligned}
$$

which gives the result. □

Proposition 7. *Under Assumptions A.1, A.2, and A.3, the date-t price of a nominal zero-coupon bond of maturity h is given by:*

$$P^{\$}_{t,h} = \exp\left(a^{\$}_h + b^{\$\prime}_h X_t + c^{\$\prime}_h Z_t + d^{\$\prime}_h vech(X_t X_t')\right), \tag{40}$$

with

$$\begin{aligned} a^{\$}_h &= -\eta_0 - \mu_{\pi,0} + a^{\$}_{h-1} + \psi^{\mathbb{Q}}_{Y,0}(u_{h-1}) \\ b^{\$}_h &= -\eta_1 + \psi^{\mathbb{Q}}_{Y,X}(u_{h-1}) \\ c^{\$}_h &= \psi^{\mathbb{Q}}_{Y,Z}(u_{h-1}) \\ d^{\$}_h &= \psi^{\mathbb{Q}}_{Y,XX}(u_{h-1}), \end{aligned}$$

where, for $h \geq 1$:

$$u_{h-1} = \left[(b^{\$}_{h-1} - \mu_{\pi,X})', (c^{\$}_{h-1} - \mu_{\pi,Z})', (d^{\$}_{h-1} - \mu_{\pi,XX})'\right]', \tag{41}$$

and $a^{\$}_0 = 0$, $b^{\$}_0 = 0$, $c^{\$}_0 = 0$, and $d^{\$}_0 = 0$, and where functions $\Psi^{\mathbb{Q}}_{Y,\bullet}$ are those given in Prop. 5.

Proof. See Appendix I.6. □

—Supplemental Appendix—

Time-varying risk aversion and inflation-consumption correlation in an equilibrium term structure model

Tilman Bletzinger, Wofgang Lemke, and Jean-Paul Renne

I Proofs

I.1 Proof of Proposition 1

When agent's preferences are as in eq. (24), the s.d.f. is given by (e.g., Piazzesi and Schneider, 2007):

$$\mathscr{M}_{t,t+1} = \delta \left(\frac{C_{t+1}}{C_t}\right)^{-1} \frac{\exp[(1-\gamma_t)u_{t+1}]}{\mathbb{E}_t(\exp[(1-\gamma_t)u_{t+1}])}. \tag{I.1}$$

Let us show that, under Assumptions A.1 and A.2, the utility is exponential affine.

Proposition 8. *Under Assumptions A.1 and A.2, the utility $U_t = C_t \exp(\mu_{u,0,t-1} + \mu'_{u,1} X_t)$, or*

$$u_t = c_t + \mu_{u,0,t-1} + \mu'_{u,1} X_t \tag{I.2}$$

satisfies eq. (24) for any X_t iff $\mu_{u,1}$ and $\mu_{u,0,t}$ are given by

$$\mu_{u,1} = \delta(I_{n_X} - \delta\Phi')^{-1}\Phi'\mu_{c,1}, \tag{I.3}$$

and

$$\mu_{u,0,t} = -\mu_{c,0} + \frac{1}{\delta}\left\{\mu_{u,0,t-1} - \frac{\delta}{1-\gamma_t}\psi_0\left[(1-\gamma_t)(\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' + I_{n_X})\mu_{c,1}\right]\right\}. \tag{I.4}$$

Proof. If u_t is given by (I.2), we have:

$$u_{t+1} = c_t + \Delta c_{t+1} + \mu_{u,0,t} + \mu'_{u,1} X_{t+1} = c_t + \mu_{u,0,t} + \mu_{c,0} + (\mu_{u,1} + \mu_{c,1})' X_{t+1}.$$

Then, for a given state vector X_t, we have:

$$\begin{aligned}
\text{eq. (24)} \quad \Leftrightarrow \quad & c_t + \mu_{u,0,t-1} + \mu'_{u,1} X_t \\
& = (1-\delta)c_t + \delta\mu_{u,0,t} + \delta\mu_{c,0} + \delta c_t + \\
& \frac{\delta}{1-\gamma_t}\left\{\psi_0\left[(1-\gamma_t)(\mu_{u,1}+\mu_{c,1})\right] + \psi_1\left[(1-\gamma_t)(\mu_{u,1}+\mu_{c,1})\right]' X_t\right\}
\end{aligned}$$

$$\Leftrightarrow \quad \mu_{u,0,t-1} - \delta\mu_{u,0,t} + \mu_{u,1}' X_t$$
$$= \delta\mu_{c,0} + \frac{\delta}{1-\gamma_t}\left\{\psi_0\left[(1-\gamma_t)(\mu_{u,1}+\mu_{c,1})\right] + \psi_1\left[(1-\gamma_t)(\mu_{u,1}+\mu_{c,1})\right]' X_t\right\}.$$

Therefore eq. (24) is satisfied for any X_t iff the following two conditions are satisfied:

$$\begin{cases} \mu_{u,0,t-1} - \delta\mu_{u,0,t} = \delta\mu_{c,0} + \frac{\delta}{1-\gamma_t}\psi_0\left[(1-\gamma_t)(\mu_{u,1}+\mu_{c,1})\right] \\ \mu_{u,1} = \frac{\delta}{1-\gamma_t}\psi_1\left[(1-\gamma_t)(\mu_{u,1}+\mu_{c,1})\right], \end{cases}$$

which leads to the result, using that $\psi_1\left[(1-\gamma_t)(\mu_{u,1}+\mu_{c,1})\right] = (1-\gamma_t)\Phi'(\mu_{u,1}+\mu_{c,1})$ under Assumption A.2. □

Using the exponential affine formulation of the utility in (I.1) leads to:

$$\begin{aligned} \log \mathscr{M}_{t,t+1} &= \log\delta - \Delta c_{t+1} + (1-\gamma_t)u_{t+1} - \log \mathbb{E}_t(\exp[(1-\gamma_t)u_{t+1}]) \\ &= \log(\delta) - \mu_{c,0} - \mu_{c,1}' X_{t+1} + (1-\gamma_t)(\mu_{c,1}+\mu_{u,1})' X_{t+1} \\ &\quad - \log \mathbb{E}_t(\exp\{(1-\gamma_t)\left[(\mu_{c,1}+\mu_{u,1})' X_{t+1}\right]\}) \\ &= \log(\delta) - \mu_{c,0} + \left[(1-\gamma_t)\mu_{u,1} - \gamma_t\mu_{c,1}\right]' X_{t+1} \\ &\quad - \log \mathbb{E}_t(\exp\{(1-\gamma_t)\left[(\mu_{c,1}+\mu_{u,1})' X_{t+1}\right]\}) \\ &= \log(\delta) - \mu_{c,0} - \psi_0\left[(1-\gamma_t)(\mu_{c,1}+\mu_{u,1})\right] + \\ &\quad \left[(1-\gamma_t)\mu_{u,1} - \gamma_t\mu_{c,1}\right]' X_{t+1} - \psi_1\left[(1-\gamma_t)(\mu_{c,1}+\mu_{u,1})\right]' X_t. \end{aligned}$$

Hence, the stochastic discount factor between dates t and $t+1$ is given by:

$$\mathscr{M}_{t,t+1} = \exp[-(\eta_{0,t}^* + \eta_1^{*\prime} X_t) + \lambda_t' X_{t+1} - \psi(\lambda_t, X_t)], \tag{I.5}$$

with

$$\begin{cases} \lambda_t &= \left\{(1-\gamma_t)\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' - \gamma_t I_{n_X}\right\}\mu_{c,1} \\ \eta_{0,t}^* &= -\log(\delta) + \mu_{c,0} + \psi_0\left[\lambda_t + \mu_{c,1}\right] - \psi_0(\lambda_t) \\ \eta_1^* &= \Phi'\mu_{c,1}. \end{cases} \tag{I.6}$$

Using $\gamma_t = \mu_{\gamma,0} + \mu_{\gamma,0}' X_t$ (Assumption A.2), we obtain:

$$\begin{aligned} \lambda_t &= \left[(1-\gamma_t)\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' - \gamma_t I_{n_X}\right]\mu_{c,1} \\ &= \left[\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' - \gamma_t(\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' + I_{n_X})\right]\mu_{c,1} \\ &= \underbrace{\left[\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' - \mu_{\gamma,0}(\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' + I_{n_X})\right]\mu_{c,1}}_{=\lambda_0} + \\ &\quad \underbrace{\left[-(\delta(I_{n_X} - \delta\Phi')^{-1}\Phi' + I_{n_X})\mu_{c,1}\mu_{\gamma,1}'\right]}_{=\lambda_1'} X_t. \end{aligned} \tag{I.7}$$

Now, let's compute $\eta_{0,t}^*$. We have (using eq. I.6):

$$\eta_{0,t}^* = -\log(\delta) + \mu_{c,0} + \psi_0(\lambda_t + \mu_{c,1}) - \psi_0(\lambda_t)$$

$$= \quad -\log(\delta) + \mu_{c,0} + \frac{1}{2}\mu_{c,1}'\Sigma\Sigma'\mu_{c,1} + \mu_{c,1}'\Sigma\Sigma'\lambda_t,$$

which proves Proposition 1.

I.2 Proof of Proposition 3

The conditional risk-neutral Laplace transform $\mathbb{E}_t^{\mathbb{Q}}(\exp(u'X_{t+1}))$ is equal to:

$$\begin{aligned}
& \mathbb{E}_t\left(\exp\left[u'X_{t+1}\right]\frac{\mathscr{M}_{t,t+1}}{\mathbb{E}_t(\mathscr{M}_{t,t+1})}\right) = \mathbb{E}_t\left(\exp\left[u'X_{t+1} + \lambda_t'\Sigma\varepsilon_{t+1} - \frac{1}{2}\lambda_t'\Sigma\Sigma'\lambda_t\right]\right) \\
= \quad & \mathbb{E}_t\left(\exp\left[u'\Phi X_t + (u+\lambda_t)'\Sigma\varepsilon_{t+1} - \frac{1}{2}\lambda_t'\Sigma\Sigma'\lambda_t\right]\right) \\
= \quad & \exp\left[u'\Phi X_t + \frac{1}{2}(u+\lambda_t)'\Sigma\Sigma'(u+\lambda_t) - \frac{1}{2}\lambda_t'\Sigma\Sigma'\lambda_t\right] \\
= \quad & \exp\left[u'\Phi X_t + \frac{1}{2}u'\Sigma\Sigma'u + u'\Sigma\Sigma'\lambda_t\right] = \exp\left[u'\Phi X_t + \frac{1}{2}u'\Sigma\Sigma'u + u'\Sigma\Sigma'(\lambda_0 + \lambda_1'X_t)\right] \\
= \quad & \exp\left[u'(\Phi + \Sigma\Sigma'\lambda_1')X_t + \frac{1}{2}u'\Sigma\Sigma'u + u'\Sigma\Sigma'\lambda_0\right],
\end{aligned}$$

which gives the result.

I.3 Proof of Proposition 2

Let us compute the Laplace transform of $Y_t = [X_t', Z_t', vec(X_tX_t')']'$. We have:

$$\begin{aligned}
& \mathbb{E}_t(\exp(u'Y_{t+1})) \\
= \quad & \mathbb{E}_t(\exp(u_X'X_{t+1} + u_Z'Z_{t+1} + u_{XX}'vec(X_{t+1}X_{t+1}'))) \\
= \quad & \mathbb{E}_t(\exp(u_X'(\Phi X_t + \Sigma\varepsilon_{t+1}) + u_Z'(\Phi_Z Z_t + \Sigma_Z(X_t)\varepsilon_{t+1}) + u_{XX}'vec((\Phi X_t + \Sigma\varepsilon_{t+1})(\Phi X_t + \Sigma\varepsilon_{t+1})'))) \\
= \quad & \exp(u_X'\Phi X_t + u_Z'\Phi_Z Z_t + u_{XX}'vec(\Phi X_t X_t'\Phi')) \times \\
& \mathbb{E}_t\left(\exp\left[(u_X'\Sigma + u_Z'\Sigma_Z(X_t) + 2u_{XX}'(\Sigma\otimes[\Phi X_t]))\varepsilon_{t+1} + u_{XX}'vec(\Sigma\varepsilon_{t+1}\varepsilon_{t+1}'\Sigma')\right]\right), \qquad \text{(I.8)}
\end{aligned}$$

where we have used, in particular, that $vec(\Phi X_t\varepsilon_{t+1}'\Sigma') = (\Sigma\otimes[\Phi X_t])\varepsilon_{t+1}$ (exploiting the properties of the vec operator, see, e.g., Proposition A.1 of Monfort et al., 2015) and also $u_{XX}'vec(\Phi X_t\varepsilon_{t+1}'\Sigma') = u_{XX}'vec(\Sigma\varepsilon_{t+1}X_t'\Phi')$ (using the following lemma, since $vec^{-1}(u_{XX})$ is assumed to be a symmetric matrix).

Lemma 1. *If matrix V is symmetric, then, for any matrix A of the same dimension, we have $vec(V)'vec(A) = vec(V)'vec(A')$.*

Proof. For any matrix A, we have $vec(A') = \Lambda_n vec(A)$, where n is the dimension of V, and where Λ_n is the commutation matrix of dimension $n^2\times n^2$ (see Lemma A.1 in Monfort et al., 2015). In particular, since matrix V is symmetric, it comes that $vec(V) = \Lambda_n vec(V)$, or $vec(V)' = vec(V)'\Lambda_n'$. Using that Λ_n is orthogonal (and therefore that $\Lambda_n'\Lambda_n = I_n$) leads to the result. □

Moreover, we have:

$$\begin{aligned} u'_{XX} vec(\Sigma \varepsilon_{t+1} \varepsilon'_{t+1} \Sigma') &= u'_{XX}(\Sigma \otimes \Sigma) vec(\varepsilon_{t+1}\varepsilon'_{t+1}) = u'_{XX}(\Sigma \otimes \Sigma)(\varepsilon_{t+1} \otimes \varepsilon_{t+1}) \\ &= (\varepsilon'_{t+1} \otimes \varepsilon'_{t+1})\{u'_{XX}(\Sigma \otimes \Sigma)\}' \\ &= \varepsilon'_{t+1} vec^{-1}\{u'_{XX}(\Sigma \otimes \Sigma)\}\varepsilon_{t+1}. \end{aligned} \quad (I.9)$$

Hence, the conditional expectation appearing at the end of (I.8) is of the form

$$\mathbb{E}_t(\exp[v(u,X_t)'\varepsilon_{t+1} + \varepsilon'_{t+1} V(u) \varepsilon_{t+1}]), \quad (I.10)$$

with

$$\begin{cases} v(u,X_t)' &= u'_X \Sigma + u'_Z \Sigma_Z(X_t) + 2u'_{XX}(\Sigma \otimes [\Phi X_t]) \\ V(u) &= vec^{-1}\{u'_{XX}(\Sigma \otimes \Sigma)\} \end{cases} \quad (I.11)$$

Let us show that $v(u,X_t)$ is affine in X_t. We have:

$$\begin{aligned} v(u,X_t) &= vec(v(u,X_t)') = vec(u'_X \Sigma + u'_Z \Sigma_Z(X_t) + 2u'_{XX}(\Sigma \otimes [\Phi X_t])) \\ &= \Sigma' u_X + (I_{n_\varepsilon} \otimes u'_Z)(\Gamma_0 + \Gamma_1 X_t) + 2(\Sigma \otimes [\Phi X_t])' u_{XX} \\ &= \Sigma' u_X + (I_{n_\varepsilon} \otimes u'_Z)(\Gamma_0 + \Gamma_1 X_t) + 2([\Phi X_t]' \otimes \Sigma') u_{XX} \\ &= \Sigma' u_X + (I_{n_\varepsilon} \otimes u'_Z)(\Gamma_0 + \Gamma_1 X_t) + 2 vec(\Sigma' vec^{-1}(u_{XX}) \Phi X_t) \\ &= \Sigma' u_X + (I_{n_\varepsilon} \otimes u'_Z)\Gamma_0 + \left[(I_{n_\varepsilon} \otimes u'_Z)\Gamma_1 + 2\Sigma' vec^{-1}(u_{XX})\Phi\right] X_t \\ &=: v_0(u) + v_1(u) X_t. \end{aligned} \quad (I.12)$$

Lemma 2. *If $\varepsilon \sim \mathcal{N}(0, I_{n_\varepsilon})$, and if v and V are, respectively, a n_ε-dimensional vector, and a $n_\varepsilon \times n_\varepsilon$ dimensional matrix, then*

$$\mathbb{E}(\exp[v'\varepsilon + \varepsilon' V \varepsilon]) = \frac{1}{|I_{n_\varepsilon} - 2V|^{1/2}} \exp\left(\frac{1}{2} v'(I_{n_\varepsilon} - 2V)^{-1} v\right).$$

Proof. See, e.g., Lemma A.2 in Dubecq et al. (2016). □

Using the previous lemma in (I.10), it comes that:

$$\begin{aligned} & \mathbb{E}_t(\exp[v(u,X_t)'\varepsilon_{t+1} + \varepsilon'_{t+1} V(u) \varepsilon_{t+1}]) \\ = & \frac{1}{|I_{n_\varepsilon} - 2V|^{1/2}} \exp\left(\frac{1}{2}(v_0 + v_1 X_t)'(I_{n_\varepsilon} - 2V)^{-1}(v_0 + v_1 X_t)\right) \\ = & \frac{1}{|I_{n_\varepsilon} - 2V|^{1/2}} \exp\left(\frac{1}{2}(v'_0(I_{n_\varepsilon} - 2V)^{-1} v_0 + X'_t v'_1 (I_{n_\varepsilon} - 2V)^{-1} v_1 X_t + 2v'_0(I_{n_\varepsilon} - 2V)^{-1} v_1 X_t)\right), \end{aligned}$$

where, for notational simplicity, we have dropped the dependency in u of v_0, v_1, and V.

Substituting in the previous expression in (I.8), we obtain:

$$\begin{aligned} & \mathbb{E}_t(\exp(u' Y_{t+1})) \\ = & \exp(u'_X \Phi X_t + u'_Z \Phi_Z Z_t + X'_t vec^{-1}\{u'_{XX}(\Phi \otimes \Phi)\} X_t \times \end{aligned}$$

$$\exp\left[-\frac{1}{2}\log|I_{n_\varepsilon}-2V|+\frac{1}{2}v_0'(I_{n_\varepsilon}-2V)^{-1}v_0+\frac{1}{2}X_t'v_1'(I_{n_\varepsilon}-2V)^{-1}v_1X_t+v_0'(I_{n_\varepsilon}-2V)^{-1}v_1X_t\right],$$

which proves Prop. 2.

I.4 Proof of Proposition 4

According to Proposition 1, the s.d.f. is a function of X_{t+1} and X_t. In this context, Lemma 1 of Monfort and Renne (2013) implies that the risk-neutral p.d.f. of Z_{t+1} and w_{t+1}, given $(X_{t+1},\underline{Y_t})$, are the same as the historical ones.

We have:

$$\begin{aligned}
Z_t &= \Phi_Z Z_{t-1}+\Sigma_Z(X_{t-1})\varepsilon_t \\
&= \Phi_Z Z_{t-1}+\Sigma_Z(X_{t-1})\Sigma^{-1}(X_t-\Phi X_{t-1}) \\
&= \Sigma_Z(X_{t-1})\Sigma^{-1}\mu^{\mathbb{Q}}+\Phi_Z Z_{t-1}+\Sigma_Z(X_{t-1})\Sigma^{-1}(\Phi^{\mathbb{Q}}-\Phi)X_{t-1}+\Sigma_Z(X_{t-1})\varepsilon_t^{\mathbb{Q}}. \qquad (I.13)
\end{aligned}$$

Note that the previous formula is also valid if the dimension of X_t is lower than that of ε_t. In this case, however, one has to replace Σ^{-1} with the Moore-Penrose inverse of Σ.

We have:

$$\begin{aligned}
\Sigma_Z(X_{t-1})\Sigma^{-1}\mu^{\mathbb{Q}} &= vec(\Sigma_Z(X_{t-1})\Sigma^{-1}\mu^{\mathbb{Q}}) \\
&= ([\Sigma^{-1}\mu^{\mathbb{Q}}]'\otimes I_{n_Z})vec(\Sigma_Z(X_{t-1})) \\
&= ([\Sigma^{-1}\mu^{\mathbb{Q}}]'\otimes I_{n_Z})(\Gamma_0+\Gamma_1X_{t-1}). \qquad (I.14)
\end{aligned}$$

Moreover, we have:

$$\begin{aligned}
\Sigma_Z(X_{t-1})\Sigma^{-1}(\Phi^{\mathbb{Q}}-\Phi)X_{t-1} &= vec(\Sigma_Z(X_{t-1})\Sigma^{-1}(\Phi^{\mathbb{Q}}-\Phi)X_{t-1}) \\
&= ([X_{t-1}'(\Sigma^{-1}(\Phi^{\mathbb{Q}}-\Phi))']\otimes I_{n_Z})(\Gamma_0+\Gamma_1X_{t-1}) \qquad (I.15)
\end{aligned}$$

Let us denote by J_i the $n_\varepsilon\times(n_\varepsilon n_Z)$ matrix that selects the following n_ε entries of a vector Γ of dimension $(n_\varepsilon n_Z)\times 1$: $\{i,n_Z+i,\dots,(n_\varepsilon-1)n_Z+i\}$. That is $J_i=I_{n_\varepsilon}\otimes e_{i,n_Z}'$, where e_{i,n_Z} is the i^{th} column of the identity matrix of dimension $n_Z\times n_Z$.

For any matrix M of dimension $n_X\times n_\varepsilon$, we have:

$$((X_{t-1}'M)\otimes I_{n_Z})\Gamma_0=\begin{bmatrix}(X_{t-1}'M)J_1\Gamma_0\\ \vdots\\ (X_{t-1}'M)J_{n_Z}\Gamma_0\end{bmatrix}=\begin{bmatrix}(\Gamma_0'J_1'M')X_{t-1}\\ \vdots\\ (\Gamma_0'J_{n_Z}'M')X_{t-1}\end{bmatrix}=\begin{bmatrix}\Gamma_0'J_1'M'\\ \vdots\\ \Gamma_0'J_{n_Z}'M'\end{bmatrix}X_{t-1} \qquad (I.16)$$

Moreover:

$$((X_{t-1}'M)\otimes I_{n_Z})\Gamma_1X_{t-1} = \begin{bmatrix}X_{t-1}'\Gamma_1'J_1'M'X_{t-1}\\ \vdots\\ X_{t-1}'\Gamma_1'J_{n_Z}'M'X_{t-1}\end{bmatrix}=\begin{bmatrix}vec(\Gamma_1'J_1'M')'vec(X_{t-1}X_{t-1}')\\ \vdots\\ vec(\Gamma_1'J_{n_Z}'M')'vec(X_{t-1}X_{t-1}')\end{bmatrix}$$

$$= \begin{bmatrix} vec(\Gamma_1' J_1' M')' \\ \vdots \\ vec(\Gamma_1' J_{n_Z}' M')' \end{bmatrix} vec(X_{t-1}X_{t-1}'). \quad \text{(I.17)}$$

Using (I.16) and (I.17) in (I.15), we obtain:

$$\begin{aligned} & \Sigma_Z(X_{t-1})\Sigma^{-1}(\Phi^{\mathbb{Q}} - \Phi)X_{t-1} \\ = & \underbrace{\begin{bmatrix} \Gamma_0' J_1' \Sigma^{-1}(\Phi^{\mathbb{Q}} - \Phi) \\ \vdots \\ \Gamma_0' J_{n_Z}' \Sigma^{-1}(\Phi^{\mathbb{Q}} - \Phi) \end{bmatrix}}_{=:\widetilde{\Gamma}_0} X_{t-1} + \underbrace{\begin{bmatrix} vec(\Gamma_1' J_1' \Sigma^{-1}(\Phi^{\mathbb{Q}} - \Phi))' \\ \vdots \\ vec(\Gamma_1' J_{n_Z}' \Sigma^{-1}(\Phi^{\mathbb{Q}} - \Phi))' \end{bmatrix}}_{=:\widetilde{\Gamma}_1} vec(X_{t-1}X_{t-1}') \quad \text{(I.18)} \end{aligned}$$

Using (I.14) and (I.18) in (I.13) leads to (35) and proves Prop. 4.

I.5 Proof of Proposition 5

We have:

$$\begin{aligned} & \mathbb{E}_t^{\mathbb{Q}}(\exp(u'Y_{t+1})) \\ = & \mathbb{E}_t^{\mathbb{Q}}(\exp(u_X'X_{t+1} + u_Z'Z_{t+1} + u_{XX}'vec[X_{t+1}X_{t+1}'])) \\ = & \mathbb{E}_t^{\mathbb{Q}}(\exp(u_X'(\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}}X_t + \Sigma\varepsilon_{t+1}^{\mathbb{Q}}) + \\ & u_Z'\{\mu_Z^{\mathbb{Q}} + \Phi_Z Z_t + \Phi_{ZX}^{\mathbb{Q}}X_t + \Phi_{ZXX}^{\mathbb{Q}}vec(X_tX_t') + \Sigma_Z(X_t)\varepsilon_{t+1}^{\mathbb{Q}}\} + u_{XX}'vec[X_{t+1}X_{t+1}'])) \\ = & \exp(u_X'\mu^{\mathbb{Q}} + u_X'\Phi^{\mathbb{Q}}X_t + u_Z'\mu_Z^{\mathbb{Q}} + u_Z'\Phi_Z Z_t + u_Z'\Phi_{ZX}^{\mathbb{Q}}X_t + u_Z'\Phi_{ZXX}^{\mathbb{Q}}vec(X_tX_t')) \times \\ & \mathbb{E}_t^{\mathbb{Q}}(\exp(u_X'\Sigma\varepsilon_{t+1}^{\mathbb{Q}} + u_Z'\Sigma_Z(X_t)\varepsilon_{t+1}^{\mathbb{Q}} + u_{XX}'vec[X_{t+1}X_{t+1}'])). \quad \text{(I.19)} \end{aligned}$$

Let us focus on the last term:

$$\begin{aligned} vec[X_{t+1}X_{t+1}'] &= vec[(\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}}X_t + \Sigma\varepsilon_{t+1}^{\mathbb{Q}})(\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}}X_t + \Sigma\varepsilon_{t+1}^{\mathbb{Q}})'] \\ &= vec\Big(\mu^{\mathbb{Q}}\mu^{\mathbb{Q}'} + \mu^{\mathbb{Q}}X_t'\Phi^{\mathbb{Q}'} + \Phi^{\mathbb{Q}}X_t\mu^{\mathbb{Q}'} + \mu^{\mathbb{Q}}\varepsilon_{t+1}^{\mathbb{Q}\,\prime}\Sigma' + \Sigma\varepsilon_{t+1}^{\mathbb{Q}}\mu^{\mathbb{Q}'} \\ &\quad + \Phi^{\mathbb{Q}}X_tX_t'\Phi^{\mathbb{Q}'} + \Phi^{\mathbb{Q}}X_t\varepsilon_{t+1}^{\mathbb{Q}\,\prime}\Sigma' + \Sigma\varepsilon_{t+1}^{\mathbb{Q}}X_t'\Phi^{\mathbb{Q}'} + \Sigma\varepsilon_{t+1}^{\mathbb{Q}}\varepsilon_{t+1}^{\mathbb{Q}\,\prime}\Sigma'\Big) \\ &= vec(\mu^{\mathbb{Q}}\mu^{\mathbb{Q}'}) + (I_{n_X^2} + \Lambda_{n_X})(\Phi^{\mathbb{Q}} \otimes \mu^{\mathbb{Q}})X_t + (\Phi^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}})vec(X_tX_t') \\ &\quad + (I_{n_X^2} + \Lambda_{n_X})(\Sigma \otimes [\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}}X_t])\varepsilon_{t+1}^{\mathbb{Q}} + (\Sigma \otimes \Sigma)vec(\varepsilon_{t+1}^{\mathbb{Q}}\varepsilon_{t+1}^{\mathbb{Q}\,\prime}), \end{aligned}$$

where Λ_{n_X} is the commutation matrix of dimension $n_X^2 \times n_X^2$ (see the proof of Lemma 1). Since $vec^{-1}(u_{XX})$ is a $n_X \times n_X$ symmetric matrix (by assumption), it comes that $u_{XX} = \Lambda_{n_X}u_{XX}$. As a result:

$$\begin{aligned} u_{XX}'vec[X_{t+1}X_{t+1}'] &= u_{XX}'vec(\mu^{\mathbb{Q}}\mu^{\mathbb{Q}'}) + 2u_{XX}'(\Phi^{\mathbb{Q}} \otimes \mu^{\mathbb{Q}})X_t + u_{XX}'(\Phi^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}})vec(X_tX_t') \\ &\quad + 2u_{XX}'(\Sigma \otimes [\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}}X_t])\varepsilon_{t+1}^{\mathbb{Q}} + u_{XX}'(\Sigma \otimes \Sigma)vec(\varepsilon_{t+1}^{\mathbb{Q}}\varepsilon_{t+1}^{\mathbb{Q}\,\prime}). \end{aligned}$$

Using the previous expression in (I.19), we obtain:

$$
\begin{aligned}
&\mathbb{E}_t^{\mathbb{Q}}(\exp(u'Y_{t+1})) = \\
&\exp\{u_X'\mu^{\mathbb{Q}} + u_{XX}' vec(\mu^{\mathbb{Q}}\mu^{\mathbb{Q}\prime}) + u_Z'\mu_Z^{\mathbb{Q}} \\
&+(u_X'\Phi^{\mathbb{Q}} + 2u_{XX}'(\Phi^{\mathbb{Q}} \otimes \mu^{\mathbb{Q}}) + u_Z'\Phi_{ZX}^{\mathbb{Q}})X_t + u_Z'\Phi_Z Z_t \\
&+(u_Z'\Phi_{ZXX}^{\mathbb{Q}} + u_{XX}'(\Phi^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}}))vec(X_t X_t')\} \times \\
&\mathbb{E}_t^{\mathbb{Q}}\left(\exp\left(\left[u_X'\Sigma + 2u_{XX}'(\Sigma \otimes \mu^{\mathbb{Q}}) + 2u_{XX}'(\Sigma \otimes [\Phi^{\mathbb{Q}}X_t]) + u_Z'\Sigma_Z(X_t)\right]\varepsilon_{t+1}^{\mathbb{Q}} + u_{XX}'(\Sigma \otimes \Sigma)vec(\varepsilon_{t+1}^{\mathbb{Q}}\varepsilon_{t+1}^{\mathbb{Q}}{}')\right)\right).
\end{aligned}
$$

The last conditional expectation is similar to that appearing in (I.8); that is, it is of the form:

$$
\mathbb{E}_t^{\mathbb{Q}}\left(v^*(u,X_t)'\varepsilon_{t+1}^{\mathbb{Q}} + \varepsilon_{t+1}^{\mathbb{Q}}{}'V(u)\varepsilon_{t+1}^{\mathbb{Q}}\right), \tag{I.20}
$$

with

$$
v^*(u,X_t)' = u_X'\Sigma + 2u_{XX}'(\Sigma \otimes \mu^{\mathbb{Q}}) + 2u_{XX}'(\Sigma \otimes [\Phi^{\mathbb{Q}}X_t]) + u_Z'\Sigma_Z(X_t),
$$

and $V(u)$ is as in (I.11).

We can proceed as in (I.12) to show that $v^*(u,X_t)$ is affine in X_t. This leads to

$$
v^*(u,X_t) = v_0^*(u) + v_1^*(u)X_t,
$$

with

$$
\left\{\begin{array}{lcl}
v_0^*(u) & = & \Sigma'u_X + 2(\Sigma' \otimes \mu^{\mathbb{Q}\prime})u_{XX} + (I_{n_\varepsilon} \otimes u_Z')\Gamma_0 \\
v_1^*(u) & = & (I_{n_\varepsilon} \otimes u_Z')\Gamma_1 + 2\Sigma' vec^{-1}(u_{XX})\Phi^{\mathbb{Q}}.
\end{array}\right. \tag{I.21}
$$

We can apply Lemma 2 to evaluate (I.20). This leads to the result presented in Prop. 5.

I.6 Proof of Prop. 7

The price of a h-period nominal bond is given by

$$
\begin{aligned}
P_{t,h}^{\$} &= \mathbb{E}_t(\mathscr{M}_{t,t+h}\exp(-\pi_{t+1} - \cdots - \pi_{t+h})) \\
&= \mathbb{E}_t\left(\frac{\mathscr{M}_{t,t+h}}{\mathbb{E}_t(\mathscr{M}_{t,t+h})}\mathbb{E}_t(\mathscr{M}_{t,t+h})\exp(-\pi_{t+1} - \cdots - \pi_{t+h})\right) \\
&= \mathbb{E}_t^{\mathbb{Q}}\left(\mathbb{E}_t(\mathscr{M}_{t,t+h})\exp(-\pi_{t+1} - \cdots - \pi_{t+h})\right) \\
&= \mathbb{E}_t^{\mathbb{Q}}(\exp(-r_t - \cdots - r_{t+h-1} - \pi_{t+1} - \cdots - \pi_{t+h})).
\end{aligned}
$$

We have:

$$
\begin{aligned}
P_{t,h+1}^{\$} &= \mathbb{E}_t^{\mathbb{Q}}\left(\exp(-r_t - \pi_{t+1})P_{t+1,h}^{\$}\right) \\
&= \mathbb{E}_t^{\mathbb{Q}}\Big(\exp\big[-(\eta_0^* + \eta_1^{*\prime}X_t) - \mu_{\pi,0} - \mu_{\pi,Z}'Z_{t+1} - \mu_{\pi,X}'X_{t+1} - \mu_{\pi,XX}'vec(X_{t+1}X_{t+1}') \\
&\quad + a_h^{\$} + b_h^{\$\prime}X_{t+1} + c_h^{\$\prime}Z_{t+1} + d_h^{\$\prime}vec(X_{t+1}X_{t+1}')\big]\Big) \\
&= \exp(-\eta_0^* - \eta_1^{*\prime}X_t - \mu_{\pi,0} + a_h^{\$}) \times
\end{aligned}
$$

$$
\begin{aligned}
& \mathbb{E}_t^{\mathbb{Q}}\left(\exp\left[(b_h^{\$}-\mu_{\pi,X})'X_{t+1}+\left(c_h^{\$}-\mu_{\pi,Z}\right)'Z_{t+1}+(d_h^{\$}-\mu_{\pi,XX})'vec(X_{t+1}X_{t+1}')\right]\right)\\
= \; & \exp(-\eta_0^*-\eta_1^{*\prime}X_t-\mu_{\pi,0}+a_h^{\$})\times\\
& \exp\left[\psi_{Y,0}^{\mathbb{Q}}(u_h)+\psi_{Y,X}^{\mathbb{Q}}(u_h)'X_t+\psi_{Y,Z}^{\mathbb{Q}}(u_h)'Z_t+\psi_{Y,XX}^{\mathbb{Q}}(u_h)'vec(X_tX_t')\right],
\end{aligned}
$$

where u_h is as defined in (41). This gives the result.

II First- and second-order moments of Y_t

In this section, we derive the first-order and second-order moments of Y_t. We consider conditional moments, say $\mathbb{E}_t(Y_{t+1})$, and unconditional moments, as $\mathbb{E}(Y_t)$.

Proposition 9. *Under Assumptions A.2 and A.3, the conditional expectation and variance of process Y_t (with $Y_t=[X_t',Z_t',vech(X_tX_t')']'$) are given by:*

$$
\mathbb{E}_t(Y_{t+1}) = \left[\frac{\partial}{\partial u}\psi_{Y,0}(u)\right]_{u=0}+\left[\frac{\partial}{\partial u}\psi_{Y,1}(u)'\right]_{u=0}Y_t \tag{II.1}
$$

$$
vec\left(\mathbb{V}ar_t(Y_{t+1})\right) = \Theta_0+\Theta_1 Y_t, \tag{II.2}
$$

where

$$
\Theta_0=\left[\frac{\partial^2}{\partial u\partial u'}\psi_{Y,0}(u)\right]_{u=0},
$$

and where Θ_1 is such that its $((i-1)n_Y+j)^{th}$ row is:

$$
\left[\frac{\partial^2}{\partial u_i\partial u_j}\psi_{Y,1}(u)'\right]_{u=0}.
$$

Proof. Since we have:

$$
\frac{\partial}{\partial u}\mathbb{E}_t(\exp(u'Y_{t+1}))=\mathbb{E}_t(Y_{t+1}\exp(u'Y_{t+1})),
$$

it comes that

$$
\left.\frac{\partial}{\partial u}\mathbb{E}_t(\exp(u'Y_{t+1}))\right|_{u=0}=\mathbb{E}_t(Y_{t+1}).
$$

Now, according to Prop. 2, we have $\mathbb{E}_t(\exp(u'Y_{t+1}))=\exp\left(\psi_{Y,0}(u)+\psi_{Y,1}(u)'Y_t\right)$. Hence, we also have:

$$
\left[\frac{\partial}{\partial u}\mathbb{E}_t(\exp(u'Y_{t+1}))\right]_{u=0}=\left[\frac{\partial}{\partial u}\psi_{Y,0}(u)\right]_{u=0}+\left[\frac{\partial}{\partial u}\psi_{Y,1}(u)'\right]_{u=0}Y_t,
$$

using, in particular, that $\psi_{Y,0}(0)=0$ and $\psi_{Y,1}(0)=0$ since $\mathbb{E}_t(\exp(0'Y_{t+1}))=1$. The variance result is obtained in a similar way. □

Corollary 1. *Under Assumptions A.2 and A.3, the dynamics of Y_t admits the following vector auto-regressive representation:*

$$Y_{t+1} = \mu_Y + \Phi_Y Y_t + \Sigma_Y^{\frac{1}{2}}(Y_t)\varepsilon_{Y,t+1}, \tag{II.3}$$

where $\varepsilon_{Y,t+1}$ is a martingale difference sequence satisfying $\mathbb{V}ar_t(\varepsilon_{Y,t+1}) = I_{n_Y}$ (identity matrix of dimension $n_Y \times n_Y$), and where

$$\mu_Y = \left[\frac{\partial}{\partial u}\psi_{Y,0}(u)\right]_{u=0}, \quad \Phi_Y = \left[\frac{\partial}{\partial u}\psi_{Y,1}(u)'\right]_{u=0},$$
$$vec(\Sigma_Y(Y_t)) := vec\left[\Sigma_Y^{\frac{1}{2}}(Y_t)\Sigma_Y^{\frac{1}{2}}(Y_t)'\right] = \Theta_0 + \Theta_1 Y_t,$$

Θ_0 and Θ_1 being given in Prop. 9.

Corollary 2. *Under Assumptions A.2 and A.3, we have:*

$$\begin{aligned}
\mathbb{E}_t(Y_{t+h}) &= (I_{n_Y} - \Phi_Y)^{-1}(I_{n_Y} - \Phi_Y^h)\mu_Y + \Phi_Y^h Y_t \\
vec\left[\mathbb{V}ar_t(Y_{t+h})\right] &= vec\left[\Sigma_Y(Y_t) + \Phi_Y\Sigma_Y(Y_t)\Phi_Y' + \dots + \Phi_Y^{h-1}\Sigma_Y(Y_t)\Phi_Y^{h-1\prime}\right] \\
&= \left(I_{n_\varepsilon^2} + \Phi_Y \otimes \Phi_Y + \dots + \Phi_Y^{h-1} \otimes \Phi_Y^{h-1}\right)(\Theta_0 + \Theta_1 Y_t) \\
&= \left(I_{n_\varepsilon^2} + \Phi_Y \otimes \Phi_Y + \dots + (\Phi_Y \otimes \Phi_Y)^{h-1}\right)(\Theta_0 + \Theta_1 Y_t) \\
&= \left(I_{n_\varepsilon^2} - \Phi_Y \otimes \Phi_Y\right)^{-1}\left(I_{n_\varepsilon} - (\Phi_Y \otimes \Phi_Y)^h\right)(\Theta_0 + \Theta_1 Y_t).
\end{aligned}$$

Corollary 3. *Under Assumptions A.2 and A.3, we have:*

$$\begin{aligned}
\mathbb{E}(Y_t) &= (I_{n_Y} - \Phi_Y)^{-1}\mu_Y \\
vec[\mathbb{V}ar(Y_t)] &= \left(I_{n_Y^2} - \Phi_Y \otimes \Phi_Y\right)^{-1} vec\left[\Sigma_Y\left\{(Id - \Phi_Y)^{-1}\mu_Y\right\}\right].
\end{aligned}$$

Corollary 4. *Under Assumptions A.2 and A.3, we have:*

$$\mathbb{C}ov(Y_t, Y_{t+h}) = \mathbb{V}ar(Y_t)(\Phi_Y^h)'.$$

Proof. We have:

$$\begin{aligned}
\mathbb{C}ov(Y_t, Y_{t+h}) &= \mathbb{E}(Y_t Y_{t+h}') - \mathbb{E}(Y_t)\mathbb{E}(Y_t)' = \mathbb{E}(Y_t \mathbb{E}_t(Y_{t+h})') - \mathbb{E}(Y_t)\mathbb{E}(Y_t)' \\
&= \mathbb{E}(Y_t[(I_{n_Y} - \Phi_Y)^{-1}(I_{n_Y} - \Phi_Y^h)\mu_Y + \Phi_Y^h Y_t]') - \mathbb{E}(Y_t)\mathbb{E}(Y_t)'
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}(Y_t\mu_Y'(I_{n_Y}-\Phi_Y^h)'(I_{n_Y}-\Phi_Y')^{-1}+Y_tY_t'(\Phi_Y^h)')-\mathbb{E}(Y_t)\mathbb{E}(Y_t)' \\
&= (I_{n_Y}-\Phi_Y)^{-1}\mu_Y\mu_Y'(I_{n_Y}-\Phi_Y^h)'(I_{n_Y}-\Phi_Y')^{-1}+\mathbb{E}(Y_tY_t')(\Phi_Y^h)'-\mathbb{E}(Y_t)\mathbb{E}(Y_t)' \\
&= (I_{n_Y}-\Phi_Y)^{-1}\mu_Y\mu_Y'(I_{n_Y}-\Phi_Y')^{-1}(I_{n_Y}-\Phi_Y^h)'+\mathbb{E}(Y_tY_t')(\Phi_Y^h)'-\mathbb{E}(Y_t)\mathbb{E}(Y_t)' \\
&= \mathbb{E}(Y_t)\mathbb{E}(Y_t)'(I_{n_Y}-\Phi_Y^h)'+\mathbb{E}(Y_tY_t')(\Phi_Y^h)'-\mathbb{E}(Y_t)\mathbb{E}(Y_t)',
\end{aligned}
$$

which gives the result. □

III Real term premiums in the Consumption Capital Asset Pricing Model

III.1 Term premiums and conditional covariances of the SDF

The term premium of maturity n is defined as the difference between the yield-to-maturity of a bond of maturity n and the one that would prevail under the expectation hypothesis. That is:

$$TP_{t,n} = -\frac{1}{n}\log \mathbb{E}_t \mathscr{M}_{t,t+n} + \frac{1}{n}\log \mathbb{E}_t \exp(-r_t - r_{t+1} - \dots - r_{t+n-1}).$$

The following proposition shows how term premiums can be expressed as a conditional covariance involving future stochastic discount factors (SDFs) and their expectations.

Proposition 10. *If the log SDF $m_{t+1} = \log(\mathscr{M}_{t,t+1})$ is Gaussian and homoskedastic, we have:*

$$\begin{aligned} TP_{t,n} &= -\frac{1}{n}\mathbb{C}ov_t(m_{t+1} - \mathbb{E}_t(m_{t+1}) + \dots + m_{t+n-1} - \mathbb{E}_{t+n-2}(m_{t+n-1}), \\ &\quad \mathbb{E}_{t+1}(m_{t+2}) + \dots + \mathbb{E}_{t+n-1}(m_{t+n})). \end{aligned}$$

Proof. We have

$$TP_{t,n} := -\frac{1}{n}\log \mathbb{E}_t(\exp(m_{t+1} + \dots + m_{t+n})) + \frac{1}{n}\log \mathbb{E}_t(\exp(-r_t - \dots - r_{t+n-1})).$$

Since m_{t+1} is Gaussian, we have

$$\exp(-r_t) = \mathbb{E}_t \exp(m_{t+1}) = \mathbb{E}_t(m_{t+1}) + \frac{1}{2}\mathbb{V}ar_t(m_{t+1}).$$

As a consequence:

$$\begin{aligned} TP_{t,n} &= -\frac{1}{n}\log \mathbb{E}_t(\exp(m_{t+1} + \dots + m_{t+n})) + \\ &\quad \frac{1}{n}\log \mathbb{E}_t\left(\exp\left(\mathbb{E}_t(m_{t+1}) + \frac{1}{2}\mathbb{V}ar_t(m_{t+1}) + \dots + \mathbb{E}_{t+n-1}(m_{t+n}) + \frac{1}{2}\mathbb{V}ar_{t+n-1}(m_{t+n})\right)\right). \end{aligned}$$

Under homoskedasticity, we have $\mathbb{V}ar_t(m_{t+1}) = \sigma_m^2$, say, for any t. This gives:

$$\begin{aligned} TP_{t,n} &= \frac{1}{2}\sigma_m^2 - \frac{1}{n}\log \mathbb{E}_t(\exp(m_{t+1} + \dots + m_{t+n})) + \\ &\quad \frac{1}{n}\log \mathbb{E}_t\left(\exp\left(\mathbb{E}_t(m_{t+1}) + \dots + \mathbb{E}_{t+n-1}(m_{t+n})\right)\right). \end{aligned}$$

Using that $m_{t+1} + \dots + m_{t+n}$ is Gaussian , we obtain:

$$TP_{t,n} = \frac{1}{2}\sigma_m^2 - \frac{1}{2n}\mathbb{V}ar_t\left[m_{t+1} + \dots + m_{t+n}\right] + \frac{1}{2n}\mathbb{V}ar_t\left[\mathbb{E}_t(m_{t+1}) + \dots + \mathbb{E}_{t+n-1}(m_{t+n})\right].$$

Let us focus on $\mathbb{V}ar_t\,[m_{t+1}+\cdots+m_{t+n}]$. Since $m_{t+1}=[m_{t+1}-\mathbb{E}_t(m_{t+1})]+\mathbb{E}_t(m_{t+1})$, we get:

$$\begin{aligned}
& \mathbb{V}ar_t\,[m_{t+1}+\cdots+m_{t+n}] \\
= \; & \mathbb{V}ar_t\,[\{m_{t+1}-\mathbb{E}_t(m_{t+1})\}+\cdots+\{m_{t+n}-\mathbb{E}_{t+n-1}(m_{t+n})\}] \\
& +\mathbb{V}ar_t\,[\mathbb{E}_t(m_{t+1})+\cdots+\mathbb{E}_{t+n-1}(m_{t+n})] \\
& +2\mathbb{C}ov_t(m_{t+1}-\mathbb{E}_t(m_{t+1})+\cdots+m_{t+n}-\mathbb{E}_{t+n-1}(m_{t+n}),\mathbb{E}_t(m_{t+1})+\cdots+\mathbb{E}_{t+n-1}(m_{t+n})).
\end{aligned}$$

Using $\mathbb{V}ar_t\,[\{m_{t+1}-\mathbb{E}_t(m_{t+1})\}+\cdots+\{m_{t+n}-\mathbb{E}_{t+n-1}(m_{t+n})\}]=n\sigma_m^2$ leads to the result. □

Proposition 10 implies in particular, for $n=2$:

$$TP_{t,2} \;=\; -\frac{1}{n}\mathbb{C}ov_t(m_{t+1},\mathbb{E}_{t+1}(m_{t+2})). \tag{III.1}$$

Hence, to have a positive two-period real term premium, we must have $\mathbb{C}ov_t(m_{t+1},\mathbb{E}_{t+1}(m_{t+2}))<0$.

III.2 A simple trend-cycle decomposition of consumption

Consider a simplified version of the model developed in the paper, with the aim of exploring analytically the slope of the term structure of real term premiums. Assume that date-t consumption, denoted by C_t, is given by:

$$C_t=C_t^*\exp(z_t),$$

where C_t^* can be interpreted as the consumption trend and z_t is its cyclical component (or output gap). Using small letters for logarithms, we get:

$$c_t=c_t^*+z_t.$$

Denoting the trend growth rate by g_t, i.e.,

$$g_t=c_t^*-c_{t-1}^*=\Delta c_t^*,$$

we obtain:

$$\Delta c_t=g_t+z_t-z_{t-1}.$$

Assume that both g_t and z_t follow auto-regressive processes of order one:

$$\begin{aligned}
g_t &= (1-\rho_g)\mu_g+\rho_g g_{t-1}+\eta_t \\
z_t &= \rho_z z_{t-1}+\nu_t,
\end{aligned}$$

where $\eta_t\sim i.i.d.\mathcal{N}(0,\sigma_g)$ and $\nu_t\sim i.i.d.\mathcal{N}(0,\sigma_z)$ (Note that what precedes implies, in particular, that $\mathbb{E}(\Delta c_t)=\mu_g$.)

For simplicity, we replace the Epstein-Zin preferences used in the paper with power-utility time-separable preferences. In that case, as is well-known, the stochastic discount factor between dates t and $t+1$ is given by:

$$\mathscr{M}_{t,t+1} \quad = \quad \delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} = \exp[\log(\delta) - \gamma\Delta c_{t+1}],$$

and, therefore, $m_{t+1} = \log \mathscr{M}_{t,t+1} = \log(\delta) - \gamma\Delta c_{t+1}$. In this context, Proposition 10 implies (this is eq. III.1):

$$TP_{t,2} = -\frac{\gamma^2}{2}\mathbb{C}ov_t[\Delta c_{t+1}, \mathbb{E}_{t+1}(\Delta c_{t+1})].$$

Since

$$\mathbb{C}ov_t[\Delta c_{t+1}, \mathbb{E}_{t+1}(\Delta c_{t+1})] = \rho_g \sigma_g^2 + (\rho_z - 1)\sigma_z^2, \tag{III.2}$$

we obtain:

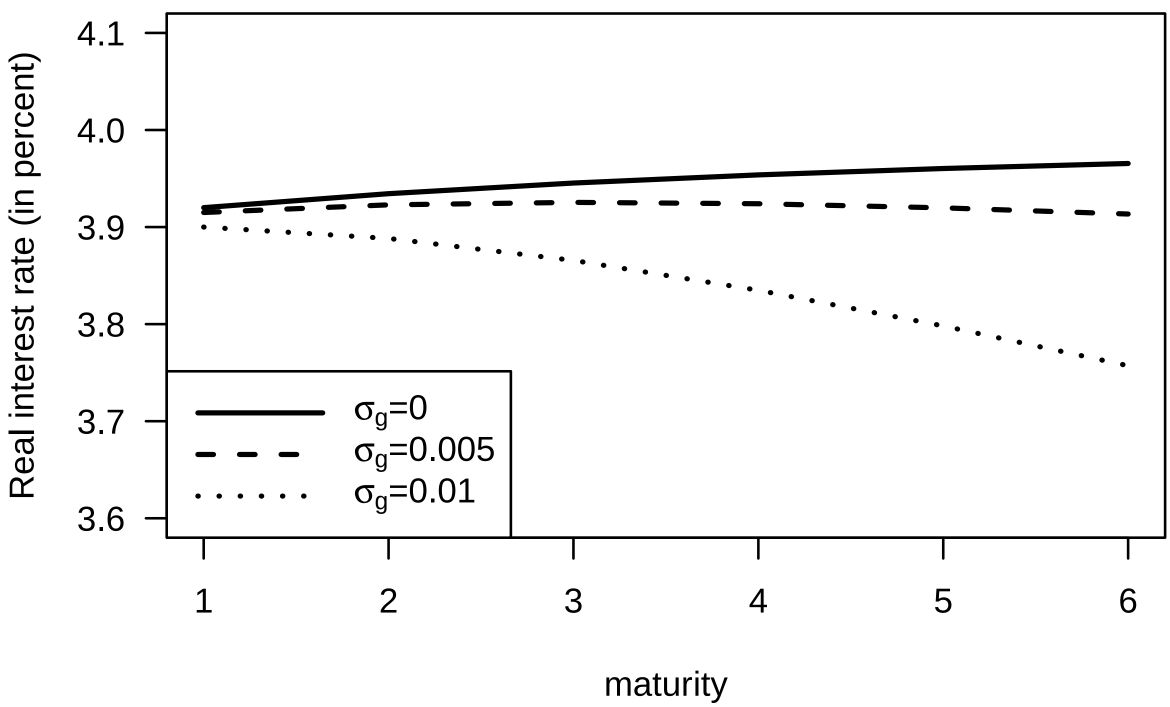
$$TP_{t,2} = \frac{\gamma^2}{2}[-\rho_g \sigma_g^2 + (1-\rho_z)\sigma_z^2],$$

which is positive if:

$$(1-\rho_z)\sigma_z^2 > \rho_g \sigma_g^2,$$

that is, if the contribution of the cyclical component dominates in $\mathbb{C}ov_t[\Delta c_{t+1}, \mathbb{E}_{t+1}(\Delta c_{t+1})]$.

Figure III.1: Average term structure of real interest rates



Notes: Term structures of real interest rates obtained for $x_t = [\mu_g, 0, 0]'$, with $\mu_g = 0.02$, $\gamma = 2$, $\delta = 1$, $\rho_g = 0.9$, $\rho_z = 0.8$, $\sigma_z = 0.02$, and different values of σ_g. The pricing formulas are those given in Proposition 11.

Figure III.1 extends this analysis for long horizons (using the pricing formulas of Proposition 10). It confirms that the real term premium real term premiums can be upward sloping if σ_g is small enough compared to σ_z (at least between horizons one and two). In particular, if $\sigma_g^2 = 0$, the term premium is positive. In that case, agents know that, if a bad state of the world materializes in the next date (i.e., $\nu_{t+1} < 0$), then the expected one-period-ahead growth, as of date $t+1$, will then be positive—since agents will then expect the output gap to close—which will translate into a higher r_{t+1}. Alternatively put, as of date t, agents know that $P_{t+1,1}$ will decrease in bad states of the world, and vice versa. This implies that a two-period bond does not hedge against bad states of the world, which generates a positive term premium.

In several papers that consider term structures of real rates in a structural framework, consumption growth is essentially based on autoregressive processes akin to g_t (this is notably the case when the model only consider a stochastic autoregressive productivity process). Hence, in these contexts, the term structure of real rates is necessarily downward sloping.

Proposition 11. *The model described in Appendix III.2 can be cast into a VAR form:*

$$x_t = \begin{bmatrix} g_t \\ z_t \\ z_{t-1} \end{bmatrix} = \underbrace{\begin{bmatrix} \mu_g(1-\rho_g) \\ 0 \\ 0 \end{bmatrix}}_{=\mu} + \underbrace{\begin{bmatrix} \rho_g & 0 & 0 \\ 0 & \rho_z & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{=\Phi} \begin{bmatrix} g_{t-1} \\ z_{t-1} \\ z_{t-2} \end{bmatrix} + \underbrace{\begin{bmatrix} \eta_t \\ \nu_t \\ 0 \end{bmatrix}}_{=\varepsilon_t}. \qquad \text{(III.3)}$$

Using the previous notations, the price of a (real) zero-coupon bond of maturity h is given by

$$P_{t,h} = \exp(a_h + b_h' x_t),$$

where

$$\left\{ \begin{array}{lcl} a_{h+1} & = & \log(\delta) + a_h + (b_h - \gamma\alpha)'\mu + \frac{1}{2}(b_h - \gamma\alpha)'\Omega(b_h - \gamma\alpha) \\ b_{h+1} & = & \Phi'(b_h - \gamma\alpha), \end{array} \right.$$

with

$$\Omega = \begin{bmatrix} \sigma_g^2 & 0 & 0 \\ 0 & \sigma_z^2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and $a_0 = 0$, $b_0 = 0$.

Proof. With the notation introduced in (III.3), we have:

$$\Delta c_t = \underbrace{\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}}_{=\alpha'} x_t,$$

and, therefore,

$$\mathscr{M}_{t,t+1} = \exp(\log(\delta) - \gamma\alpha' x_{t+1}).$$

Hence:

$$\begin{array}{rcl} P_{t,h+1} & = & \mathbb{E}_t(\mathscr{M}_{t,t+1} P_{t+1,h}) = \mathbb{E}_t(\exp[\log(\delta) - \gamma\alpha' x_{t+1} + a_h + b_h' x_{t+1}]) \\ & = & \mathbb{E}_t(\exp[\log(\delta) + a_h + (b_h - \gamma\alpha) x_{t+1}]) \end{array}$$

$$
\begin{aligned}
&= \mathbb{E}_t(\exp[\log(\delta) + a_h + (b_h - \gamma\alpha)'(\mu + \Phi x_t + \varepsilon_{t+1})]) \\
&= \exp[\log(\delta) + a_h + (b_h - \gamma\alpha)'(\mu + \Phi x_t)]\mathbb{E}_t(\exp[(b_h - \gamma\alpha)'\varepsilon_{t+1}]) \\
&= \exp\left[\log(\delta) + a_h + (b_h - \gamma\alpha)'(\mu + \Phi x_t) + \frac{1}{2}(b_h - \gamma\alpha)'\Omega(b_h - \gamma\alpha)\right],
\end{aligned}
$$

which leads to the result. □

III.3 About the implications of the trend/cycle specification on consumption moments

This appendix discusses some implications that the trend-cycle representation of consumption growth has on the conditional moments of consumption.

We have seen that, in the context of the model described in Appendix III, we have (this is eq. III.2):

$$\mathbb{C}ov_t[\Delta c_{t+1}, \mathbb{E}_{t+1}(\Delta c_{t+1})] = \rho_g \sigma_g^2 + (\rho_z - 1)\sigma_z^2.$$

Besides, since

$$\mathbb{V}ar_t(\Delta c_{t+1}) = \sigma_g^2 + \sigma_z^2, \quad \text{and} \quad \mathbb{V}ar_t(\mathbb{E}_{t+1}\Delta c_{t+2}) = \rho_g^2\sigma_g^2 + (\rho_z - 1)^2\sigma_z^2,$$

it comes that:

$$\mathbb{C}orr_t[\Delta c_{t+1}, \mathbb{E}_{t+1}(\Delta c_{t+1})] = \frac{\rho_g \sigma_g^2 + (\rho_z - 1)\sigma_z^2}{\sqrt{\sigma_g^2 + \sigma_z^2}\sqrt{\rho_g^2\sigma_g^2 + (1-\rho_z)^2\sigma_z^2}}.$$

It can be noteed that this correlation takes extreme values when $\sigma_g = 0$, in which case it is equal to -1, and when $\sigma_z = 0$, in which case it is equal to 1. If direct data-based counterparts of this *conditional* correlation were available, one could determine whether one of these two extreme representations has to be discarded; but this is not the case. One can nevertheless exploit surveys to determine *unconditional* correlations between output growth and expected output growth. The model-implied unconditional correlation is given by:

$$\mathbb{C}orr(\Delta c_{t+1}, \mathbb{E}_{t+1}(\Delta c_{t+1})) = \frac{\frac{\rho_g}{1-\rho_g^2}\sigma_g^2 - \frac{1-\rho_z}{1+\rho_z}\sigma_z^2}{\sqrt{\frac{1}{1-\rho_g^2}\sigma_g^2 + \frac{2}{1+\rho_z}\sigma_z^2}\sqrt{\frac{\rho_g^2}{1-\rho_g^2}\sigma_g^2 + \frac{1-\rho_z}{1+\rho_z}\sigma_z^2}}.$$

It can be seen that, when $\sigma_z = 0$, this correlation is still extreme, as it is equal to one. But it is not extreme for $\sigma_g = 0$. In the latter case, we have:

$$\mathbb{C}orr(\Delta c_{t+1}, \mathbb{E}_{t+1}(\Delta c_{t+1})) = -\sqrt{\frac{1-\rho_z}{2}}.$$

Let us consider empirical estimates of this correlation. For that, we use the real GDP series available on the FRED database (ticker GDPC1), as well as the mean GDP forecasts of professional forecasters extracted from the website of the Philadelphia Federal Reserve Bank. Using these data, we can compute the empirical correlation between the quarterly output growth ($\log(Y_t/Y_{t-1})$) and expected output growth ($\mathbb{E}_t(\log(Y_{t+1}/Y_t))$). We also consider an annual version, where we compute the correlation between $\log(Y_t/Y_{t-4})$ and $\mathbb{E}_t(\log(Y_{t+4}/Y_t))$. Table III.1 shows the resulting correlations for two periods: a long historical period (1968-2024), and a shorter one (1994-2024). The last row of the table reports the model-implied equivalent correlations (using the model presented in Section 2 and whose specification is detailed in Table 2).

Table III.1: Unconditional correlations between current and expected output growth

Period	**Quarterly**	**Annual**
1968Q4-2024Q1	27%	17%
1994Q1-2024Q1	−4%	7%
Model	−14%	−7%

Notes: This table shows the correlations between output growth ($\log(Y_t/Y_{t-k})$) and expected output growth ($\mathbb{E}_t(\log(Y_{t+k}/Y_t))$). For column "Quarterly", we use $k = 1$, and $k = 4$ for column "Annual". Real GDP comes from the FRED database (ticker GDPC1); GDP forecasts are extracted from the Philadelphia Federal Reserve Bank website. The last line of the table reports the model-implied equivalent correlations; the model is the one presented in Section 2, whose specification is detailed in Table 2.

The low but rather positive (for three subsamples) empirical correlation between GDP growth and expected future GDP growth is not perfectly in line with the present framework. However, it strongly rejects those frameworks featuring a pure auto-regressive consumption growth process, i.e. a model without output gap ($\sigma_z = 0$), for which the correlation between GDP growth and expected GDP growth is equal to one.

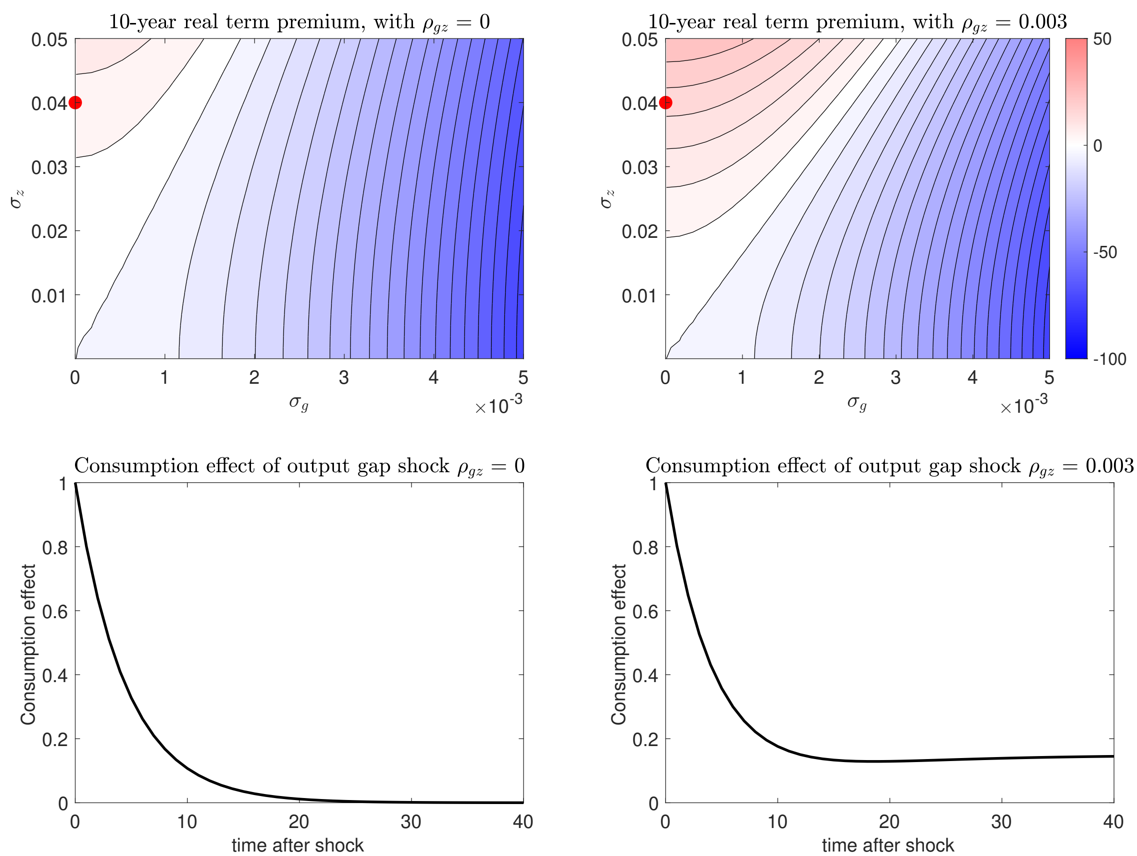
III.4 The relevance of hysteresis effects for the term structure of real rates

Section 2.5 discussed the main model ingredients for generating an upward-sloping real term structure. This annex analyses the amplifying hysteresis channel in more depth.

In our specifications, hysteresis effects are introduced through parameter ρ_{gz} (see eq. 1): if $\rho_{gz} > 0$, periods of negative output gap (z_t) imply reductions in the trend of consumption growth (g_t). Hence, a recession ($z_t < 0$) is a bad state of the world for two compounded reasons: (i) by definition, consumption is low—below it trend—when $z_t < 0$, and (ii) when $\rho_{gz} > 0$, the fact that $z_t < 0$ reduces expected trend growth rate. This is illustrated by the lower plots of Figure III.2, which compare the response of the consumption level (c_t) to increases in z_t in two situations: no hysteresis effect ($\rho_{gz} = 0$) for the left plot and existence of an hysteresis effect ($\rho_{gz} > 0$) for the right plot. The key difference is that while the effect completely dies out when $\rho_{gz} = 0$, it is not the case when $\rho_{gz} > 0$. Consequently, for a given state of recession, hysteresis effects worsen consumption prospects, leading to higher risk prices which, in turn, amplifies forward premiums.

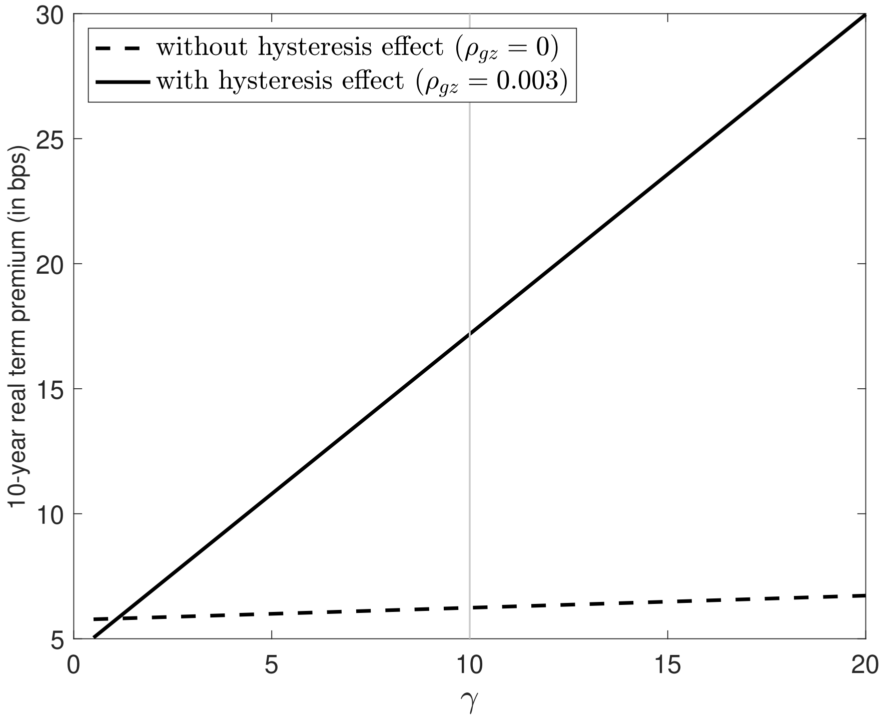
The upper plots of Figure III.2 illustrate the influence of hysteresis effects on real term premiums. Term premiums appear to be larger when $\rho_{gz} > 0$ (i.e., with hysteresis effect, right plot), than when $\rho_{gz} = 0$ (left plot). Figure III.3 shows how the term premiums depend on the coefficient of risk aversion, for the values of σ_z and σ_g indicated on Figure III.2 with red dots. Real term premiums appear to be more sensitive to the coefficient of risk aversion with hysteresis effects.

Figure III.2: Real term premium and hysteresis effect



Notes: This figure illustrates the influence of the permanent and transitory and permanent consumption shocks, as well as the hysteresis effect, on the real term premium. The real term premium is defined in (23)); it is given by $\mathbb{E}(r_{t,40} - r_{t,1})$; it is also the average slope of the term structure of real rates. Only the real part of the model is concerned (i.e., eqs. 1, 2, 12), agents feature Epstein-Zin preferences (see Subsection 2.2), with a constant coefficient of risk aversion. The left plots correspond to the case where $\rho_{gz} = 0$—the situation with no hysteresis effects; by contrast, there is an hysteresis effect in the model underlying the right plots. The upper plots show how the real term premium depends on σ_g and σ_z, that are the respective standard deviations of the shocks affecting the persistent component of consumption *growth* (g_t) and the transitory component of the cyclical component of consumption *level* (z_t). The lower plots show the impulse response functions of (log) consumption c_t to a unit increase in z_t; the bottom-right plot shows that, in a context of hysteresis, these shocks ($\varepsilon_{z,t}$) have a permanent effect on consumption. The upper plots show that real term premium positively depend on σ_z and negatively on σ_g. The top-right plot also shows that, when σ_g is low, the hysteresis effects allow to generate higher (positive) real term premiums. The model parameterization is as follows: $\rho_z = 0.8$, $\rho_g = 0.9$, $\mu_c = 2\%$, $\gamma_t = \mu_{\gamma,0} = 10$. The red dots indicate the values of σ_z and σ_g used in Figure III.3.

Figure III.3: Real term premium, hysteresis effect, and risk aversion



Notes: This figure illustrates the influence of the risk aversion coefficient on the term premium. The real term premium is defined in (23)); it is given by $\mathbb{E}(r_{t,40} - r_{t,1})$; it is also the average slope of the term structure of real rates. We consider two models: one is with hysteresis effects ($\rho_{gz} > 0$, solid line) and the other is without hysteresis effects ($\rho_{gz} = 0$, dotted line). Only the real part of the model is concerned (i.e., eqs. 1, 2, 12), agents feature Epstein-Zin preferences (see Subsection 2.2), with a constant coefficient of risk aversion γ. We consider different values of the coefficient of risk aversion (x axis). The model parameterization is as follows: $\rho_z = 0.8$, $\rho_g = 0.9$, $\mu_c = 2\%$, $\gamma_t = \mu_{\gamma,0} = 10$. The values of σ_z and σ_g are those indicated by red dots in Figure III.2; for the model with hysteresis effects: $\rho_{gz} = 0.003$. The vertical bar indicates the value of the coefficient of risk aversion used in Figure III.2.